# On the Computational Power of Iterative Auctions II: Ascending Auctions

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**Abstract**

We embark on a systematic analysis of the power and limitations of iterative ascending-price combinatorial auctions. We prove a large number of results showing the boundaries of what can be achieved by different models of ascending auctions: item prices vs. bundle prices, anonymous prices vs. personalized prices, deterministic vs. non-deterministic, ascending vs. descending, preference elicitation vs. full elicitation, adaptive vs. non-adaptive, and single trajectory vs. multi trajectory. Two of our main results show that neither ascending item-price auctions nor ascending anonymous bundle-price auctions can determine the optimal allocation among general valuations. This justifies the use of personalized bundle prices in iterative combinatorial auctions like the FCC spectrum auctions.

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# 1 Introduction

In combinatorial auctions, a set of heterogeneous indivisible goods are for sale. Each bidder may have a different value for every subset of these items. The goal is to partition the items among the bidders such that the social welfare (the total value of the bidders for the bundles they receive) is maximized. Selling each item in a separate auction may ignore economic relations between the goods, and thus may incur a severe economic inefficiency: the goods may be *complements*, where the value of a bundle is greater than the sum of the values of its parts (e.g., a left shoe and a right shoe together worth more than the sum of the singleton values) or the opposite case of *substitutes* items (e.g., an airline ticket and a train ticket to the same destination).

The computational hardness of combinatorial auctions is two sided: the bidders have to communicate private information with an exponential size. Moreover, even if we assume that each bidder is interested in a single bundle only, finding the optimal allocation is NP-complete [17]. While the latter problem can be solved in practice for rather large-scale problems, the communication problem seems to be harder to solve – and this problem is the focus of this work. Many *iterative* auctions have been suggested in the literature (see, e.g., the survey [22]) to overcome this problem: the bidders do not reveal their whole preferences, but only respond to queries from the auctioneer. Researchers have studied different models for iterative combinatorial auctions, and in a companion paper [7], we study some of these models and try to characterize the differences between them, and specifically, the power of different types of queries. In this work, we focus on *ascending auctions* – iterative auctions in which the published prices can only rise in time.

We find the study of ascending auctions appealing for various reasons. First, ascending auctions are widely used in many real-life settings from the FCC spectrum auctions [10] to almost any e-commerce website (e.g., [2, 1]). Actually, this is maybe the most straightforward way to sell items: ask the bidders what would they like to buy under certain prices, and increase the prices of over-demanded goods. Ascending auctions are also considered more intuitive for many bidders, and are believed to increase the "trust" of the bidders in the auctioneer, as they see the result gradually emerging from the bidders' responses. Ascending auctions also have other desired economic properties, e.g., they incur smaller information revelation (consider, for example, English auctions vs. second-price sealed bid auctions).

The main distinction between the different kinds of ascending auctions is in their pricing methods: some models present only *item prices (linear prices)*, where the price of each bundle is the sum of the prices of the items in this bundle. Others choose to use *bundle prices (non-linear prices)* in which each bundle may have a price of its own. In some auctions, the same *anonymous* prices are presented to all bidders, while in auctions that use *non-anonymous* (discriminatory) prices, each bidder is presented with a personalized set of prices. Note that for ascending auctions, anonymous prices is a severe restriction: it has a single ascending trajectory of prices, where in non-anonymous auctions we may have a different ascending trajectory of prices per each bidder. Each of the many different proposed auctions has some subset of the above properties. Diagram 1 (in Appendix F) summarizes the basic "classes" of auctions implied by combinations of the above properties and classifies some of the auctions proposed in the literature according to this classification.

In this work, we try to systematically analyze what do the differences between these models of auctions mean. We try to answer the following questions: *(i) Which models of ascending auctions can find the optimal allocation, and for which classes of valuations? (ii) In cases where the optimal allocation cannot be determined by ascending auctions, how well can such auctions approximate the social welfare? (iii) How do the different models for ascending auctions compare? Are some models computationally stronger than others?*

The starting point of our treatment of ascending combinatorial auctions is the existence in the literature of several such auctions that always obtain an optimal allocation. In particular the "iBundle(3)" auction of [24] and the "Proxy auction" of [3]. Interestingly, both auctions use non-anonymous bundle prices. In this work, we show that it is no coincidence: any other type of ascending auctions cannot always find the optimal allocation or even a reasonable approximation.

We study ascending auctions as a tool for solving the *"preference elicitation"* problem: eliciting partial information about the bidders' preferences in order to determine the optimal allocation. Note that this

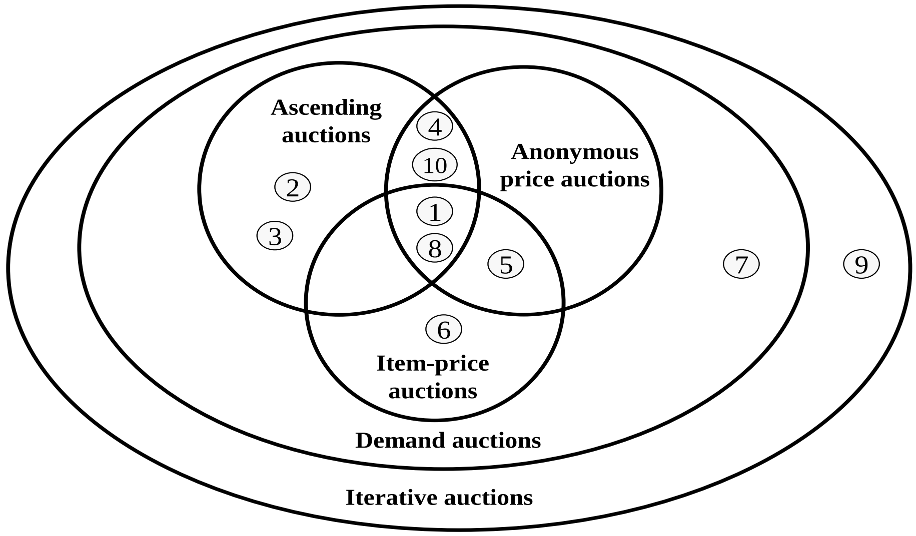


Figure 1: The diagram classifies the following auctions according to their properties:
**(1)** Demange, Gale & Sotomayor's [9] adaptation for Kelso & Crawford's [14] auction for substitutes valuations.
**(2)** The Proxy Auction [3] by Ausubel & Milgrom.
**(3)** iBundle(3) by Parkes & Ungar [21].
**(4)** The iBundle(2) anonymous auction by Parkes & Ungar [24].
**(5)** Our descending adaptation for the 2-approximation for submodular valuations by [16] (see Subsection 4.1).
**(6)** Ausubel's [4] truthful auction for substitutes valuations.
**(7)** The adaptation by Nisan & Segal [20] of the $O(\sqrt{m})$ approximation by [17].
**(8)** The auction by Bartal, Gonen & Nisan [5] for duplicate-item auctions.
**(9)** The auction by Zinkevich, Blum & Sandholm [28] for Read-Once formulae.
**(10)** The AkBA Auction by Wurman & Wellman [27].

is a different problem than *"full elicitation"* in which we aim to exactly learn the bidders' valuations for every possible bundle. Many times, the information needed for preference elicitation is much smaller than what is needed for full elicitation. Several recent papers study the connection between these two problems ([28, 6, 15, 26]).

Ascending auctions have been extensively studied in the literature (see the recent survey by Parkes [22]). Most of this work presented 'upper bounds', i.e., proposed mechanisms with ascending prices and analyzed their properties. A result which is closer in spirit to ours, is by Gul and Stacchetti [12], who showed that no item-price ascending auction can always determine the VCG prices, even for substitutes valuations.[1] Our framework is more general than the traditional line of research that concentrates on the *final* allocation and payments and in particular, on reaching 'Walrasian equilibria' or 'Competitive equilibria'. A Walrasian equilibrium[2] is known to exist in the case of Substitutes valuations, and is known to be impossible for any wider class of valuations [11]. This does not rule out other allocations by ascending auctions: in this paper we view the auctions as a computational process where the outcome - both the allocation and the payments - can be determined according to all the data elicited throughout the auction; This general framework strengthens our negative results.[3]

We do not treat here any incentive issues (i.e., we assume that the bidders, or an oracle on their behalf, respond truthfully to all queries), and we assume that the parties are computationally unbounded (e.g., they can solve NP-hard problems). However, these two strong assumptions *strengthen* our hardness results.

This paper is composed of a relatively large number of results. Next, we try to put these results in their context, and we distinct item-price and bundle-price ascending auctions.

[1] We further discuss this result in Section 3.3.

[2] A Walrasian equilibrium is vector of item prices for which all the items are sold when each bidder receives a bundle in his demand set.

[3] In few recent auction designs (e.g., [4, 18]) the payments are not necessarily the final prices of the auctions.

### 1.1 Ascending Item-Price Auctions

It is well known that the item-price ascending auctions of Kelso and Crawford [14] and its variants [9, 11] find the optimal allocation as long as all players' valuations have the *substitutes* property. The obvious question is whether the optimal allocation can be found for a larger class of valuations. Our first result here is a strong negative result, with several extensions:

- We present a simple 2-item 2-player problem where no ascending item-price auction can find the optimal allocation. The same proof proves a similar impossibility result for other models of auctions. This is in contrast to both the power of bundle-price ascending auctions (see below) and to the power of general item-price demand queries [7], both of which can always find the optimal allocation and in fact even provide full preference elicitation.
- We show that at least $k-1$ ascending item-price trajectories are needed to elicit XOR formulae with $k$ terms. This result is in some sense tight, since we show that any $k$-term XOR formula can be fully elicited by $k-1$ *non-deterministic* (i.e., when some exogenous "teacher" instructs the auctioneer on how to increase the prices)[4] ascending auctions. Moreover, eliciting some classes of valuations requires an *exponential* number of ascending item-price trajectories.
- We observe that item-price ascending auctions and iterative auctions that are limited to a *polynomial* number of queries (of any kind, not necessarily ascending) are incomparable in their power: ascending auctions, *with small enough increments*, can elicit the preferences in cases where any polynomial number of queries cannot.

Motivated by several recent papers that studied the relation betweem eliciting and fully-eliciting the preferences in combinatorial auctions (e.g., [6, 15]), we explore the difference between these problems in the context of ascending auctions. We show that although a single ascending auction can determine the optimal allocation among any number of bidders with substitutes valuations, it cannot *fully-elicit* such a valuation of a single bidder. While [16] show that the set of substitutes valuations has measure zero in the space of general valuations, its dimension is not known, and in particular it is still open whether a polynomial amount of information suffices to describe a substitutes valuation. While our result may be a small step in that direction, we note that our impossibility result also holds for valuations in the class OXS defined by [16], valuations that we are able to show have a compact representation.

We also give several results separating the power of different models for ascending combinatorial auctions: we prove, not surprisingly, that *adaptive* ascending auctions are more powerful than *oblivious* ascending auctions and that *non-deterministic* ascending auctions are more powerful than *deterministic* ascending auctions. We also compare different kinds of *non-anonymous* auctions (e.g., simultaneous or sequential), and observe that anonymous bundle-price auctions and non-anonymous item-price auctions are incomparable in their power. Finally, motivated by Dutch auctions, we consider *descending auctions*, and how they compare to ascending ones. We show classes of valuations that can be elicited by ascending item-price auctions but not by descending item-price auctions, and vice versa.

### 1.2 Ascending Bundle-Price Auctions

All known ascending bundle-price auctions that are able to find the optimal allocation between general valuations use non-anonymous prices. Anonymous ascending-price auctions are only known to be able to find the optimal allocation among super-additive valuations or few other simple classes ([23]). We show that this is no mistake:

- No ascending auction with anonymous prices can find the optimal allocation between general valuations. This bound is regardless of the running time, and it also holds for descending auctions and non-deterministic auctions.
- We strengthen this result significantly by showing that *anonymous* ascending auctions cannot produce a better than $O(\sqrt{m})$ approximation ($m$ is the number of items for sale) – the approximation

[4] Non-deterministic computation is widely used in CS and also in economics (e.g, a Walrasian equilibrium or [25]). In some settings, deterministic and non-deterministic models have equal power (e.g., computation with finite automata).

ratio that can be achieved with a polynomial number of queries ([17, 20]) and in particular, with a polynomial number of item-price demand queries ([7]).[5]

Finally, we study the performance of the existing computationally-efficient ascending auctions. These protocols ([24, 3]) require exponential time in the worst case, and this is unavoidable as shown by [20]. However, we also observe that these auctions, as well as the whole class of similar ascending bundle-price auctions, require an exponential time even for simple additive valuations. This *is* avoidable and indeed the ascending item-price auctions of [14] can find the optimal allocation for these valuations with polynomial communication.

**Open questions:** we do not know how to achieve any non-trivial approximations for unrestricted valuations by either anonymous bundle-price auctions or ascending item-price auctions. We also do not currently have a lower bound for the approximation achievable by *non-anonymous* item-price ascending auctions. In addition, we leave the exact characterization of the valuations that are elicitable by each type of ascending auctions as an open question.

**The organization of the paper**: Section 2 describes our model. Section 3 discusses the power of item-price ascending auction, and Section 4 compares different models of such auctions. In Section 5 we study the power of bundle-price ascending auctions.

# 2 The Model

## 2.1 Discrete Auctions for Continuous Values

Our model aims to capture iterative auctions that operate on real-valued valuations. There is a slight technical difficulty here in bridging the gap between the discrete nature of an iterative auction, and the continuous nature of the valuations. This is exactly the same problem as in modeling a simple English auction. There are three standard formal ways to model it:

1. Model the auction as a continuous process and study its trajectory in time. For example, the so-called Japanese auction is basically a continuous model of an English model.[6]
2. Model the auction as discrete and the valuations as continuously valued. In this case we introduce a parameter $\epsilon$ and usually require the auction to produce results that are $\epsilon$-close to optimal.
3. Model the valuations as discrete. In this case we assume that all valuations are integer multiples of some small fixed quantity $\delta$, e.g., 1 penny. All communication in this case is then naturally finite. We can assume without loss of generality that $\delta = 1$ hence all valuations are integral.

In this paper we use the latter formulation and assume that all values are multiples of some $\delta$. Almost all (if not all) of our results can be translated to the other two models with little effort.

## 2.2 Valuations

We wish to sell a set $M$ of $m$ indivisible items among $n$ bidders. Each bidder $i$ has a valuation function $v_i : 2^M \rightarrow \{0, \delta, 2\delta, ..., L\}$ that attaches a value $v_i(S)$ for any bundle $S \subseteq M$. All values are multiples of some $\delta > 0$. We assume two standard assumptions for combinatorial auctions: (i) Free disposal, i.e., if $S \subset T$ then $v_i(S) \leq v_i(T)$. (ii) Normalization, i.e., $v_i(\emptyset) = 0$ for every bidder $i$.

We will mention the following classes of valuations:

- A valuation is called *sub-modular* if for all sets of items $A$ and $B$ we have that $v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$.
- A valuation is called *super-additive* if for all disjoint sets of items $A$ and $B$ we have that $v(A \cup B) \geq v(A) + v(B)$.

[5]The same bound clearly holds for (anonymous) item-price ascending auctions since such auctions can be simulated by anonymous bundle-price ascending auctions.

[6]Another similar model is the "moving knives" model in the cake-cutting literature.

- A valuation is called a *k-bundle XOR* if it can be represented as a XOR combination of at most $k$ atomic bids [19], i.e., if there are at most $k$ bundles $S_i$ and prices $p_i$ such that for all $S$, $v(S) = max_{i|S \supseteq S_i} p_i$. Such valuations will be denoted by $v = (S_1 : p_1) \oplus (S_2 : p_2) \oplus \ldots \oplus (S_k : p_k)$.[7]

### 2.3 Iterative Auctions

The goal of the auctioneer is to maximize the *social welfare* (or the *economic efficiency)*, that is, to find a partition (or *allocation*) $S_1, ..., S_n$ of the items that maximizes $\sum_{i=1}^{n} v_i(S_i)$. Note that our goal is not to maximize the seller's revenue, but to make the "society" as content as we can. The auctioneer repeatedly queries the bidders in order to gain sufficient information for determining the optimal allocation.

A protocol (or an "auction" or an "algorithm") is the method the auctioneer determines which queries to ask. The queries may be determined adaptively, i.e., as a function of the history of queries and responses, or obliviously - where all the queries are predefined. We say that a protocol has a polynomial running-time (or is computationally efficient) if the number of queries asked is always polynomial in the number of bidders $n$, the number of items $m$ and in $\frac{L}{\delta}$, where $L$ is the maximal possible value.[8] Note that we do not require our protocols to run in time polynomial in $\log L$, as would be required had we wanted to run in time that is polynomial in the representation of each value. This is because that ascending auctions can usually not achieve such running times (consider even the English auction on a single item). Most of the auctions we present may be adapted to run in time polynomial in $\log L$, using a binary-search-like procedure, losing their ascending nature.

We say than an auction *elicits* some class $V$ of valuations, if it determines the optimal allocation for any profile of valuations drawn from $V$; We say that an auction achieves a *c-approximation* for the optimal welfare $OPT$, if it achieves a welfare of at least $\frac{OPT}{c}$ for any profile of valuations. We say that an auction *fully elicits* some class of valuations $V$, if it can fully learn any valuation $v \in V$ (i.e., learn $v(S)$ for every $S$).

### 2.4 Ascending Auctions

Maybe the simplest kind of query is the "value query" - given a bundle $S$, the bidder reports his value $v_i(S)$ for this bundle. In this work we concentrate on auctions that use a different type of query - the "demand query". In a demand query, given a set of prices for the bundles, the bidder responds with one of his most desired bundles, i.e., a bundle that maximizes his (quasi linear) utility $v(S) - p(S)$.[9] In a companion paper ([7]) we give a detailed discussion of the power of demand queries and how do they compare to other types of valuations. Specifically, [7] shows that demand queries are significantly more powerful than value queries. Ascending auctions are iterative auctions with non-decreasing prices:

**Definition 1.** In an *ascending auction*, the prices in the queries to the same bidder can only increase in time. Formally, let $p$ be a query made for bidder $i$, and $q$ be a query made for bidder $i$ at a later stage in the protocol. Then for all sets $S$, $q(S) \geq p(S)$. A similar variant, which we also study and that is also common in real life, is *descending auctions*, in which prices can only decrease in time.

Note that the term "ascending auction" refers to an auction with a single ascending trajectory of prices. It may be useful to define multi-trajectory ascending auctions, in which the prices maybe reset to zero a number of times (see, e.g., [4]). We consider two main restrictions on the types of allowed demand queries:

**Item Prices:** The prices in each query are given by prices $p_j$ *for each item* $j$. The price of a set $S$ is additive: $p(S) = \sum_{j \in S} p_j$. We say that an auction uses *bundle prices* if each bundle $S$ may have a different price $p(S)$ (which is not necessarily the sum of the prices of the items in $S$).

**Anonymous prices:** The prices seen by the bidders at any stage in the auction are the same, i.e. whenever a query is made to some bidder, the same query is also made to all other bidders (with

[7] For example, $v = (abcd : 5) \oplus (ab : 3) \oplus (c : 4)$ denotes the XOR valuation with the terms $abcd, ab, c$ and prices $5, 3, 4$ respectively. For this valuation, $v(abcd) = 5$, $v(abd) = 3$, $v(abc) = 4$.

[8] Note that the private data of a bidder is exponential in $m$.

[9] All of our impossibility results hold for any consistent tie-breaking rule by the bidders or by the auctioneer.

the prices unchanged). In *non-anonymous* auctions, each bidder $i$ has personalized prices denoted by $p_i(S)$.[10] In this paper, all auctions are anonymous unless otherwise specified.

Note that although we assume that the valuations are multiples of some $\delta$, we allow the demand queries to use arbitrary numbers in $\Re_+$. That is, we assume that the increment $\epsilon$ may be significantly smaller than $\delta$. All our hardness results hold for any $\epsilon$, even for continuous price increments.

# 3 Item-Price Ascending Auctions

In this section we characterize the power of ascending item-price auctions. We first show that this power is not trivial: such auctions can in general elicit an exponential amount of information. On the other hand, we show that the optimal allocation cannot always be determined by a single ascending auction, and in some cases, nor by an exponential number of ascending-price trajectories Finally, we separate the power of different models of ascending auctions. All missing proofs from this section appear in Appendix A.

## 3.1 The Power of Item-Price Ascending Auctions

We first show that if small enough increments are allowed, a single ascending trajectory of item-prices can elicit preferences that cannot be elicited with polynomial communication.

**Theorem 1.** *Some classes of valuations can be elicited by item-price ascending auctions, but cannot be elicited by a polynomial number of queries of any kind.*

*Proof.* Consider two bidders with $v(S) = 1$ if $|S| > \frac{n}{2}$, $v(S) = 0$ if $|S| < \frac{n}{2}$ and every $S$ such that $|S| = \frac{n}{2}$ has an unknown value from $\{0, 1\}$. Due to [20], determining the optimal allocation here requires exponential communication in the worst case. Nevertheless, we show (in Appendix A) that an item-price ascending auction can do it, as long as it can use exponentially small increments. □

We now describe another positive result for the power of item-price ascending auctions. In a companion paper [7], we show that a value query can be simulated with a (truly) polynomial number of item-price demand queries. Here, we show that every value query can be simulated by a (pseudo) polynomial number of *ascending* item-price demand queries. (In the next subsection, we show that we cannot always simulate *a pair* of value queries using a single item-price ascending auction.) In Appendix E, we show that we can simulate other natural or useful queries using item-price ascending auctions.

**Proposition 1.** *A value query can be simulated by an item-price ascending auction. This simulation requires a polynomial number of queries.*

Actually, the proof for Proposition 1 proves a stronger useful result regarding the information elicited by iterative auctions. It says that in *any* iterative auction in which the changes of prices are small enough in each stage ("pseudo-continuous" auctions), the value of all bundles demanded along the auction can be computed. The basic idea is that when the bidder moves from demanding some bundle $T_i$ to demanding another bundle $T_{i+1}$, there is a point in which she is indifferent between these two bundles. Thus, knowing the value of some demanded bundle (e.g., the empty set) enables computing the values of all other demanded bundles.

We say that an auction is *"pseudo-continuous"*, if it only uses demand queries, and in each step, the price of at most one item is changed by $\epsilon$ (for some $\epsilon \in (0, \delta]$) with respect to the previous query.

**Proposition 2.** *Consider any pseudo-continuous auction (not necessarily ascending), in which bidder $i$ demands the empty set at least once along the auction. Then, the value of every bundle demanded by bidder $i$ throughout the auction can be calculated at the end of the auction.*

[10] Note that a non-anonymous auction can clearly be simulated by $n$ parallel anonymous auctions.

| | v(ab) | v(a) | v(b) |
|---|---|---|---|
| **Bidder 1** | 2 | $\alpha \in (0,1)$ | $\beta \in (0,1)$ |
| **Bidder 2** | 2 | 2 | 2 |

Figure 2: No item-price ascending auction can determine the optimal allocation for this class of valuations.

### 3.2 Limitations of Item-Price Ascending Auctions

Although demand queries can solve any combinatorial auction problem [7], when they are restricted to be ascending some classes of valuations cannot be learned nor elicited. An example for such valuations is given in Figure 2.

**Theorem 2.** *There are classes of valuations that cannot be elicited nor fully elicited by any item-price ascending auction.*

*Proof.* Let bidder 1 have the valuation described in the first row of Figure 2, where $\alpha$ and $\beta$ are unknown values in $(0, 1)$. First, we prove that this class cannot be fully elicited by a single ascending auction. Specifically, an ascending auction cannot reveal the values of both $\alpha$ and $\beta$.

As long as $p_a$ and $p_b$ are both below 1, the bidder will always demand the whole bundle $ab$: her utility from $ab$ is strictly greater than the utility from either $a$ or $b$ separately. For example, we show that $u_1(ab) > u_1(a)$:

$$u_1(ab) \;=\; 2-(p_a+p_b) \;=\; 1-p_a+1-p_b \;>\; v_A(a)-p_a+1-p_b \;>\; u_1(a)$$

Thus, in order to gain any information about $\alpha$ or $\beta$, the price of one of the items should become at least 1, w.l.o.g. $p_a \geq 1$. But then, the bundle $a$ will not be demanded by bidder 1 throughout the auction, thus no information at all will be gained about $\alpha$.

Now, assume that bidder 2 is known to have the valuation described in the second row of Figure 2. The optimal allocation depends on whether $\alpha$ is greater than $\beta$ (in bidder 1's valuation), and we proved that an ascending auction cannot determine this. □

Note that an immediate conclusion is that this impossibility result also holds for item-price *descending* auctions and even for the wider classes of *non-deterministic* item-price auctions (in which some exogenous data can tell us how to increase the prices) and *non-anonymous* ascending item-price auctions.

### 3.3 Limitations of Multi-Trajectory Ascending Auctions

According to Theorem 2, no ascending item-price auction can always elicit the preferences (we prove a similar result for bundle prices in section 5). But can two ascending trajectories do the job? Or a polynomial number of ascending trajectories? We give negative answers for such suggestions.

We define a *k-trajectory ascending auction* as a demand-query iterative auction in which the demand queries can be partitioned to $k$ sets of queries, where the prices published in each set only increase in time. Note that we use a general definition; It allows the trajectories to run in parallel or sequentially, and to use information elicited in some trajectories for determining the future queries in other trajectories.

The power of multiple-trajectory auctions can be demonstrated by the negative result of Gul and Stacchetti [12] who showed that even for an auction among substitutes valuations, an anonymous ascending item-price auction cannot compute VCG prices for all players. Ausubel [4] overcame this impossibility result and designed auctions that do compute VCG prices by organizing the auction as a sequence of $n+1$ ascending auctions. Here, we prove that one cannot elicit XOR valuations with $k$ terms by less than $k-1$ ascending trajectories. On the other hand, we show that an $XOR$ formula can be fully elicited by $k-1$ non-deterministic ascending auctions (or by $k-1$ deterministic ascending auctions if the auctioneer knows the atomic bundles).[11]

[11] This result actually separates the power of deterministic and non-deterministic iterative auctions: Our proof shows that a non-deterministic iterative auction can elicit the $k$-term XOR valuations with a polynomial number of demand queries, and [6] show that this elicitation must take an exponential number of demand queries.

**Proposition 3.** *XOR valuations with k terms cannot be elicited (or fully elicited) by any (k-2)-trajectory item-price ascending auction, even when the atomic bundles are known to the elicitor. However, these valuations can be elicited (and fully elicited) by (k-1)-trajectory non-deterministic non-anonymous item-price ascending auctions.*

Moreover, an exponential number of trajectories is required for eliciting some classes of valuations:

**Proposition 4.** *Elicitation and full-elicitation of some classes of valuations cannot be done by any k-trajectory item-price ascending auction, where* $k = o(2^m)$.

*Proof.* Consider the following class of valuations: For $|S| < \frac{m}{2}$, $v(S) = 0$ and for $|S| > \frac{m}{2}$, $v(S) = 2$. Every bundle $S$ of size $\frac{m}{2}$ has some unknown value in $(0, 1)$. In Appendix A we show that a single ascending auction can reveal the value of at most one bundle of size $\frac{n}{2}$. □

# 4 Separating the Various Models of Ascending Auctions

Various models for ascending auctions have been suggested in the literature. In this section, we compare the power of the different models. We also discuss the difference between preference elicitation and full elicitation. As mentioned, all auctions are considered anonymous and deterministic, unless specified otherwise. All missing proofs from this section are in Appendix B.

## 4.1 Separation Results

***Ascending vs. Descending Auctions***: We begin the discussion of the relation between ascending auctions and descending auctions with an example. We present (in Figure 5 in Appendix D.1) a simple item-price descending auction that guarantees at least half of the optimal efficiency for submodular valuations.[12] However, we are not familiar with any *ascending* auction that guarantees a similar fraction of the efficiency. This raises a more general question: can ascending auctions solve any combinatorial-auction problem that is solvable using a descending auction (and vice versa)? We give negative answers to these questions. The idea behind the proofs is that the information that the auctioneer can get "for free" at the beginning of each type of auction is different.[13]

**Proposition 5.** *There are classes that cannot be elicited (fully elicited) with ascending item-price auctions, but can be elicited (resp. fully elicited) with a descending item-price auction.*

**Proposition 6.** *There are classes that cannot be elicited (fully elicited) by item-price descending auctions, but can be elicited (resp. fully elicited) by item-price ascending auctions.*

***Deterministic vs. Non-Deterministic Auctions***: Non-deterministic ascending auctions can be viewed as auctions where some benevolent teacher that has complete information guides the auctioneer on how she should raise the prices. That is, preference elicitation can be done by a non-deterministic ascending auction, if there is *some* ascending trajectory that elicits enough information for determining the optimal allocation (and verifying that it is indeed optimal). We show that non-deterministic ascending auctions are more powerful than deterministic ascending auctions:

**Proposition 7.** *Some classes can be elicited (fully elicited) by an item-price non-deterministic ascending auction, but cannot be elicited (resp. fully elicited) by item-price deterministic ascending auctions.*

***Anonymous vs. Non-Anonymous Auctions***: As will be shown in Section 5, the power of anonymous and non-anonymous bundle-price ascending auctions differs significantly. Here, we show that a difference also exists for item-price ascending auctions.

[12] In Appendix D.1, we prove that this descending auction actually implements the algorithm by Lehmann, Lehmann and Nisan [16].

[13] In ascending auctions, the auctioneer can reveal the most valuable bundle (besides $M$) before she starts raising the prices, thus she can use this information for adaptively choose the subsequent queries. In descending auctions, one can easily find the bundle with the highest *average per-item* price, keeping all other bundles with non-positive utilities, and use this information in the adaptive price change.

**Proposition 8.** *Some classes cannot be elicited by anonymous item-price ascending auctions, but can be elicited by a non-anonymous item-price ascending auction.*

***Sequential vs. Simultaneous Auctions***: A non-anonymous auction is called *simultaneous* if at each stage, the price of some item is raised by $\epsilon$ for *every* bidder. The auctioneer can use the information gathered until each stage, in all the personalized trajectories, to determine the next queries.

A non-anonymous auction is called *sequential* if the auctioneer performs an auction for each bidder separately, in sequential order. The auctioneer can determine the next query based on the information gathered in the trajectories completed so far and on the history of the current trajectory.

**Proposition 9.** *There are classes that cannot be elicited by simultaneous non-anonymous item-price ascending auctions, but can be elicited by a sequential non-anonymous item-price ascending auction.*

***Adaptive vs. Oblivious Auctions:*** If the auctioneer determines the queries regardless of the bidders' responses (i.e., the queries are predefined) we say that the auction is *oblivious*. Otherwise, the auction is *adaptive*. We prove that an adaptive behaviour of the auctioneer may be beneficial.

**Proposition 10.** *There are classes that cannot be elicited (fully elicited) using oblivious item-price ascending auctions, but can be elicited (resp. fully elicited) by an adaptive item-price ascending auction.*

### 4.2 Preference Elicitation vs. Full Elicitation

Preference elicitation and full elicitation are closely related problems. If full elicitation is "easy" (e.g., in polynomial time) then clearly elicitation is also easy (by a *non-anonymous* auction, simply by learning all the valuations separately[14]). On the other hand, there are examples where preference elicitation is considered "easy" but learning is hard (typically, elicitation requires smaller amount of information; some examples can be found in [6]).

The *tatonnement* algorithms by [14, 9, 11] (see Figure 6 in Appendix F) end up with the optimal allocation for substitutes valuations.[15] We prove that we cannot fully elicit substitutes valuations (or even their sub-class of $OXS$ valuations defined in [16]), even for a single bidder, by an item-price ascending auction (although the optimal allocation can be found by an ascending auction for any number of bidders!).

**Theorem 3.** *Substitute valuations cannot be fully elicited by ascending item-price auctions. Moreover, they cannot be fully elicited by any* $\frac{m}{2}$ *ascending trajectories (*$m > 3$*).*

Whether substitutes valuations have a compact representation (i.e., polynomial in the number of goods) is an important open question. As a step in this direction, we show in Appendix B.7 that its sub-class of $OXS$ valuations does have a compact representation: every $OXS$ valuation can be represented by at most $m^2$ values.[16]

## 5 Bundle-Price Ascending Auctions

All the ascending auctions in the literature that are proved to find the optimal allocation for unrestricted valuations are non-anonymous bundle-price auctions (iBundle(3) by Parkes and Ungar [24] and the "Proxy Auction" by Ausubel and Milgrom [3]). Yet, several *anonymous* ascending auctions have been suggested (e.g., AkBA [27], [13] and iBundle(2) [24]). In this section, we prove that anonymous bundle-price ascending auctions achieve poor results in the worst-case. We also show that the family of non-anonymous bundle-price ascending auctions can run exponentially slower than simple item-price ascending auctions.

[14]Note that an anonymous ascending auction cannot necessarily elicit a class that can be fully elicited by an ascending auction.

[15]Substitute valuations are defined in Appendix D.2). For completeness, we give in Appendix F a proof for the efficiency of this auction for valuations with the substitutes property.

[16]A unit-demand valuation is an XOR valuation in which all the atomic bundles are singletons. OXS valuations can be interpreted as an aggregation ("OR") of any number of unit-demand bidders. See Appendix D.2 for formal definition.

| | | | |
|---|---|---|---|
| **Bidder 1** | $v_1(ac) = 2$ | $v_1(bd) = 2$ | $v_1(cd) = \alpha \in (0, 1)$ |
| **Bidder 2** | $v_2(ab) = 2$ | $v_2(cd) = 2$ | $v_2(bd) = \beta \in (0, 1)$ |

Figure 3: Anonymous ascending bundle-price auctions cannot determine the optimal allocation for this class of valuations.

### 5.1 Limitations of Anonymous Bundle-Price Ascending Auctions

We present a class of valuations that cannot be elicited by anonymous bundle-price ascending auctions. These valuations are described in Figure 3. The basic idea: for determining some unknown value of one bidder we must raise a price of a bundle that should be demanded by the other bidder in the future.

**Theorem 4.** *Some classes of valuations cannot be elicited by anonymous bundle-price ascending auctions.*

*Proof.* Consider a pair of XOR valuations as described in Figure 3. For finding the optimal allocation we must know which value is greater between $\alpha$ and $\beta$.[17] However, we cannot learn the value of *both* $\alpha$ and $\beta$ by a single ascending trajectory: assume w.l.o.g. that bidder 1 demands $cd$ before bidder 2 demands $bd$ (no information will be elicited if none of these happens). In this case, the price for $bd$ must be greater than 1 (otherwise, bidder 1 prefers $bd$ to $cd$). Thus, bidder 2 will never demand the bundle $bd$, and no information will be elicited about $\beta$. □

The valuations described in the proof of Theorem 4 can be easily elicited by a non-anonymous *item-price* ascending auction. On the other hand, the valuations in Figure 2 can be easily elicited by anonymous bundle-price ascending auction. We conclude that the power of these two families of ascending auctions is incomparable.

We strengthen the impossibility result above by showing that anonymous bundle-price auctions cannot even achieve better than a $\min\{O(n), O(\sqrt{m})\}$-approximation for the social welfare. This approximation ratio can be achieved with polynomial communication, and specifically with a polynomial number of item-price demand queries [7].[18]

**Theorem 5.** *An anonymous bundle-price ascending auction cannot guarantee better than a* $\min\{\frac{n}{2}, \frac{\sqrt{m}}{2}\}$*-approximation for the optimal welfare.*

*Proof.* (sketch) We construct a combinatorial design that creates a hard-to-elicit class for $n$ bidders and $n^2$ items. We need $n^2$ distinct bundles with the following properties: for each bidder, we define a partition $S^i = (S^i_1, ..., S^i_n)$ of the $n^2$ items to $n$ bundles, such that any two bundles from different partitions intersect. (we give an explicit construction for this design using the properties of linear functions over finite fields.)

Now, each bidder has a value of 2 for all the bundles in his partition, and a value of either 0 or $1-\delta$ for the bundles in the other partitions. Since bundles from different partitions intersect, we cannot give a value of 2 for more than a single bidder. We show that any anonymous bundle-price auction will not achieve more than a welfare of 2 in the worst case, where the optimal welfare may be arbitrarily close to $n+1$. (Full proof is in Appendix C.2). □

### 5.2 Bundle Prices vs. Item Prices

The core of the auctions in [24, 3] is the scheme described in Figure 4 (in the spirit of [22]) for auctions with non-anonymous bundle prices. Auctions from this scheme end up with the optimal allocation for any class of valuations. We denote this family of ascending auctions as NBEA auctions[19].

[17] If $\alpha > \beta$, the optimal allocation will allocate $cd$ to bidder 1 and $ab$ to bidder 2. Otherwise, we give $bd$ to bidder 2 and $ac$ to bidder 1. Note that both bidders cannot gain a value of 2 in the same allocation, due to the intersections of the high-valued bundles.

[18] Note that bundle-price queries may use exponential communication, thus the lower bound of [20] does not hold.

[19] *N*on-anonymous *B*undle-price economically *E*fficient *A*scending auctions. For completeness, we give in Appendix F a simple proof for the efficiency (up to an $\epsilon$) of auctions of this scheme .

**Non-anonymous Bundle-Price Economically-Efficient Ascending Auctions:**
**Initialization:** All prices are initialized to zero (non-anonymous bundle prices).
**Repeat:** Each bidder submits a bundle that maximizes his utility under his current personalized prices. The auctioneer calculates a provisional allocation that maximizes his *revenue* under the current prices. The prices of bundles that were demanded by losing bidders are increased by $\epsilon$.
**Finally:** Terminate when the provisional allocation assigns to each bidder the bundle he demanded.

Figure 4: Auctions from this family (denoted by NBEA auctions) are known to achieve the optimal welfare.

NBEA auctions can elicit $k$-term XOR valuations by a polynomial (in $k$) number of steps (see Proposition 12 in Appendix C), although the elicitation of such valuations may require an exponential number of item-price queries ([6]), and item-price ascending auctions cannot do it at all (Theorem 2). Nevertheless, we show that NBEA auctions (and in particular, iBundle(3) and the "proxy" auction) are sometimes inferior to simple item-price demand auctions. This may justify the use of hybrid auctions that use both linear and non-linear prices (e.g., the clock-proxy auction [8]). We show that auctions from this family may use an exponential number of queries even for determining the optimal allocation among two bidders with additive valuations[20], where such valuations can be elicited by a simple item-price ascending auction. We actually prove this property for a wider class of auctions we call *conservative auctions*. We also observe that in conservative auctions, allowing the bidders to submit all the bundles in their demand sets ensures that the auction runs a polynomial number of steps – if $L$ is not too high (but with exponential communication, of course). (See proof in Appendix C.)

An ascending auction is called *conservative* if it is non-anonymous, uses bundle prices initialized to zero and at every stage the auctioneer can only raise prices of bundles demanded by the bidders until this stage. In addition, each bidder can only receive bundles he demanded during the auction.[21]

**Proposition 11.** *If every bidder demands a single bundle in each step of the auction, conservative auctions may run for an exponential number of steps even for additive valuations. If the bidders are allowed to submit all the bundles in their demand sets in each step, then conservative auctions can run in a polynomial number of steps for any profile of valuations, as long as the maximal valuation $L$ is polynomial in $m, n$ and $\frac{1}{\delta}$.*

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# A Item-Price Ascending Auctions - Missing Proofs

**Proof for Theorem 1:**

*Proof.* Consider two bidders with $v(S) = 1$ for $|S| > \frac{n}{2}$, $v(S) = 0$ for $|S| < \frac{n}{2}$ and every $S$ such that $|S| = \frac{n}{2}$ has an unknown value from $\{0, 1\}$. Due to [20], determining the optimal allocation for these two bidders requires an exponential communication in the worst case. Nevertheless, we show that an item-price ascending auction can do it, as long as it can use sufficiently small increments.

Let $S$ be some bundle of $\frac{n}{2}$ items. First, we observe that when the price of all the items in $S$ equals $\lambda$, and the price of all items in $M \setminus S$ is $\lambda + \epsilon$ ($\lambda, \epsilon > 0$ and $\lambda + \epsilon \leq 1/m$, the bidder will demand the bundle $S$ if and only if $v(S) = 1$: the price of the bundle $S$ is $|S| \cdot \lambda$ but the price of any other bundle with size of at least $\frac{n}{2}$ will be greater and no subset of $S$ has a non-zero value. (When $v(S) = 0$ the bidder will clearly demand some bigger bundle.).

Thus, the following ascending auction will fully elicit the valuations: let $\{S_1, S_2, ...\}$ be a list of all the bundles of size $\frac{n}{2}$. The auction starts from zero prices, then increments the prices of the items in $M \setminus S_1$ by some small $\epsilon$ (see below). We observed that at this point we can determine if $v(S_1) = 1$ for both bidders. Then, we increase the prices of the items in $S_1$ by $\epsilon$ such that all the prices become equal. We repeat a similar process for $S_2$,$S_3$ and so on. This way we can enumerate on all the bundles of size $\frac{n}{2}$, until we determine all their values. Clearly, $\epsilon$ should be exponentially small, and indeed it is easy to see that $\epsilon = \Theta(2^{-(m+\log m)})$ is sufficient. □

**Proof for Proposition 1:**

*Proof.* The following ascending auction learns the value of a given bundle $S$:
<u>*Initialization*</u>*: start with a zero price for every item in $S$, and price of $L$ for every item in $M \setminus S$.*
<u>*Repeat:*</u> *raise the price of each item in $S$ by $\epsilon = \delta$ in turn, in a round-robin fashion.*
<u>*Finally*</u>*: Terminate when the bidder demands the empty set.*

We claim that from the information elicited by this ascending auction we can calculate $v(S)$. In the initial stage, the bidder demands $S$ or another bundle with the same value (due to the free-disposal assumption). Let $T_1, ..., T_k$ be the bundles demanded by the bidder in the order they were demanded (bundles might repeat). We know that $T_1 = S$ and $T_k = \emptyset$. We prove that if we know the value of some bundle $T_{i+1}$ we can calculate $v(T_i)$. Thus, since we know that $v(\emptyset) = 0$, $v(S)$ can be calculated (by induction).

If we could raise the prices continuously, the proof would be very easy. Since prices are increased in a discrete manner, we should be more careful. In particular, we assume that we know the tie breaking rule of the bidder (i.e., which bundle he would demand if he had few bundles with the highest utility).

Let $\overrightarrow{p}$ be the smallest vector of prices in which the bidders demands $T_{i+1}$, and this happened after we raised the price of some item $k$ by $\epsilon$. If $T_i$ has a priority over $T_{i+1}$ in the bidder's tie breaking rule, then we know that his utility from $T_{i+1}$ under the prices $\overrightarrow{p}$ is $\epsilon$-higher than his utility from $T_i$ (clearly, $k \in T_i$ but $k \notin T_{i+1}$, otherwise the demand change wouldn't happen). Thus,

$$v(T_{i+1}) - p(T_{i+1}) = v(T_i) - p(T_i) + \epsilon$$

Since the prices are known, we can calculate $v(T_i)$ from $v(T_{i+1})$. Similarly, if the bidder's tie breaking rule favors $T_{i+1}$, then the utilities at this point should be equal, i.e.,

$$v(T_{i+1}) - p(T_{i+1}) = v(T_i) - p(T_i)$$

and we can similarly calculate $v(T_i)$ from $v(T_{i+1})$. The total running time is at most $\frac{m \cdot L}{\delta}$. □

**Proof for Proposition 2:**

*Proof.* The same proof for Proposition 1 holds here: note that in that proof the way the prices were increased had no significance, as long as the bidders eventually demanded the empty set. In addition, it did not matter in the proof if the prices were increased or decreased, as long as the changes were small (pseudo-continuously). □

**Proof for Proposition 3:**

*Proof.* Let $A_i = M \setminus \{i\}$ (for $i = 1, ..., m$), and consider the following XOR valuation with $m$ terms:

$$v = (A_1 : 2) \oplus (A_2 : \alpha_2) \ldots \oplus (A_m : \alpha_m)$$

Where $\alpha_2, ..., \alpha_m$ are unknown values between $(0,1)$. Let $A_j$ be the first bundle among $\{A_2, ..., A_m\}$ to be demanded by the bidder. Let $\overrightarrow{p}$ be the prices in which $A_j$ is demanded. Since $v(A_1) - p(A_1) \leq v(A_j) - p(A_j)$ and due to the linear prices, it follows that $p_j - p_1 \geq v(A_1) - v(A_j) > 1$, and thus $p_j > 1$. Therefore, none of the other bundles from $A_2, ..., A_m$ will be demanded in this ascending trajectory, so revealing each value of the bundle $A_2, ..., A_m$ requires a separate ascending auction. Note that this proof also holds for non-deterministic auctions or descending auctions, and even when the auctioneer knows the bundles composing the XOR formula.

Next, we show that if one knows the bundles in the XOR formula, she can determine their value with $k-1$ (deterministic) ascending auction. It will follow that $k-1$ *non-deterministic* ascending auctions suffice for learning any XOR valuation with $k$ terms. Let $A_1, ..., A_k$ be the terms of the XOR valuation, ordered w.l.o.g. according to their values (and then according to the bidder's tie breaking rule, thus $A_1$ is the first bundle to be demanded). Since $A_1$ has the highest valuation, and due to the free-disposal assumption, there is $j \neq 1$ such that $A_1 \setminus A_j \neq \emptyset$. We raise the price of some item $x$ in $A_1 \setminus A_j$ until $A_j$ is demanded, and then we can simulate $v(A_j)$ with a similar procedure as in Proposition 1. Due to Proposition 2, since $A_1$ was demanded, the value $v(A_1)$ can also be calculated.

In addition, we will calculate $v(A_l)$ for all $l \neq 1, j$ in separate ascending auctions as described in Proposition 1. In total, we run no more than $k-1$ ascending trajectories, which revealed all the bundles' values.

One more issue that should be addressed is that these $k-1$ trajectories must reveal enough information for eliciting the preferences. For that, the information must suffice for knowing that each bundle $A_i$ is minimal, i.e., there is no $A_j \subset A_i$ such that $v(A_j) = v(A_i)$. And indeed, if the ascending auction simulating $v(A_i)$ described in the proof for Proposition 1 increases the prices with a small $\epsilon$ ($\epsilon < \frac{\delta}{m}$) in a round-robin fashion, then if such $A_j$ exists, it will be discovered by the protocol.

Consider a bidder with a valuation from the above class, and consider a second bidder that has a value of 2 for every singleton, except $\{1\}$. The optimal allocation must find the most valuated singleton of bidder 1, but learning all the singleton values of bidder 1 takes at least $k-1$ ascending trajectories, thus one cannot determine the optimal allocation for these valuations. □

**Proof for Proposition 4:**

*Proof.* Consider the following class of valuations: For $|S| < \frac{m}{2}$, $v(S) = 0$ and for $|S| > \frac{m}{2}$, $v(S) = 2$. Every bundle $S$ of size $\frac{m}{2}$ has some unknown value between $(0,1)$. We show that fully eliciting such valuations requires an exponential number of ascending *trajectories*. We first prove the following claim:

*Claim* 1. Along every ascending trajectory, the bidder demands at most one bundle of size $\frac{m}{2}$.

*Proof.* Assume that under the price vector $\overrightarrow{p}$, the bidder demands some bundle $S$, $|S| = \frac{m}{2}$, for the first time. Thus, for any item $x \in M \setminus S$, the bidder weakly prefers the bundle $S$ over the bundle $\{S \cup x\}$, i.e., $v(S \cup x) - p(S \cup x) \leq v(S) - p(S)$. Since the prices are linear, it follows that: $p_x \geq v(S \cup x) - v(S) > 2 - v(S) > 1$. Thus, the price of *any item* in $M \setminus S$ is greater then 1. It follows that the bidder will never demand any bundle of size $\frac{m}{2}$ containing an item from $M \setminus S$. The only bundle of size $\frac{m}{2}$ that does not contain any item from $M \setminus S$ is $S$. □

Since there are $\Omega(2^n)$ such bundles, an exponential number of ascending trajectories is needed for the full elicitation of these valuations.

Now, consider a second player that has a valuation of 2 for any bundle of size $\frac{m}{2}$ or higher. The optimal allocation will clearly allocate the $\frac{m}{2}$ items that bidder 1 valuates the most to bidder 1, and the other $\frac{m}{2}$ items to the second bidder. However, learning all these values requires an exponential number of ascending trajectories. □

# B Separating Item-Price Ascending auctions - proofs

## B.1 Ascending Auctions vs. Descending Auctions

**Proof for Proposition 5:**

*Proof.* Consider a class of XOR valuations with 3 terms over 3 items $\{a, b, c\}$ of the following form:

$$v = (abc : 2) \oplus (x : 1) \oplus (y : \alpha)$$

where $\alpha$ is an unknown value between $(0, \frac{1}{2})$ and $x, y$ are some unknown consecutive items (i.e., $(x, y) \in \{(a, b), (b, c), (c, a)\}$).

We first show that the following descending auction can fully elicit this class of valuations: start with a price of 1 for all items. Then, the bidder will demand $\{x\}$. The identity of $x$ also reveals item $y$, thus we can decrease $p_y$ until $\{y\}$ is demanded, thus $\alpha$ is revealed. (Note that the bidder will not demand the bundle $\{abc\}$, since its price stays above 2.)

Next, we show that no ascending auction can learn this class of valuation: starting from zero prices, a change in the demand can occur only if either $p_x + p_w \geq 1$ or $p_y + p_w \geq 1$ (where $w$ is the 3rd item besides $x, y$). Thus, information will be gained only after the auctioneer arbitrarily increases the price of one of the items above $\frac{1}{2}$ (without any input received until this point in the auction). If this item is item $y$, the bidder will clearly never demand the bundle $\{y\}$, and thus the value of $\alpha$ will not be revealed.

We now prove similar result for the elicitation of this class of valuations. Consider the class of valuations described above. We first show that a descending auction can always find the optimal allocation for any number of bidders (we assume $n > 2$, otherwise allocating all the items to any bidder is optimal). We start with a price of 1 for all items. Under these prices, each bidder will demand the bundle with a value of 1 (i.e., $\{x\}$). If we have three bidders that demand three different items, we allocate each of the item to the bidder that demands it, and it is clearly the optimal allocation. If all the bidders demand the same item $x$, then the optimal allocation is achieved by allocating all items to one of the bidders. If the bidders demand two different items, these items must be consecutive, w.l.o.g. $a$ and $b$. An optimal allocation will allocate $a, b$ to bidders that demands them, and $c$ to the bidder with the highest value for it. This bidder is the first bidder to demand $c$ when we decrease $p_c$. (The price of the whole bundle will still be greater than 2, so no bidder will demand it.)

Next, we show that no item-price ascending auction can determine the optimal allocation for the above class. Consider three bidders drawn from the class of valuations above. One bidder have a valuation of 1 for some item $x_1$, and the two other bidders have a valuation of 1 for the subsequent item $x_2$, and let $x_3$ be the 3rd item ($x_1, x_2, x_3 \in \{a, b, c\}$). The optimal allocation will allocate $x_1$ to the first bidder, $x_3$ to the bidder with the highest value for it (i.e., with the highest $\alpha$), and $x_2$ to the remaining bidder. For any set of item prices which are all smaller than $\frac{1}{2}$, there will be no change in the demand of the bidders and no information about the identity of these items will be extracted. Thus, in order to elicit any information, the auctioneer must arbitrarily raise one of the prices of the items above $\frac{1}{2}$. However, if this item turns out to be $x_3$, then no bidder will demand the bundle $\{x_3\}$, and the auctioneer cannot know who is the bidder with the highest valuation for this bundle. Therefore, the allocation may not be optimal. □

**Proof for Proposition 6:**

*Proof.* Consider 4 items $\{x_1, x_2, x_3, x_4\}$, and a class of valuations of the form $v = (x_{i-1}x_ix_{i+1} : 2.5) \oplus (x_{i-1}x_i : 2) \oplus (x_i : \alpha)$ *or* of the form $v = (x_{i-1}x_ix_{i+1} : 2.5) \oplus (x_ix_{i+1} : 2) \oplus (x_i : \alpha)$, where $\alpha$ is an unknown value between $(0, 1)$, the index $i$ is unknown and the indices are cyclic.

An ascending auction can learn such valuations: for zero prices, the bidder demands $x_{i-1}x_ix_{i+1}$, revealing the index $i$.[22]

After increasing the price of $x_{i-1}$ and $x_{i+1}$ to $\frac{1}{2}$, the bidder demands the 2-item bundle. Now, we raise the prices of all items except $x_i$ to $L$. Finally, we raise the price of $x_i$ and the price where the bidder stops demanding $\{x_i\}$ is $\alpha$.

Next, we show that no item price descending auction can fully elicit this valuation. No information can be elicited as long as all prices are greater or equal to 1, since no bundle will be demanded (except, maybe, the 2-item bundle from which the identity of the item $x_i$ is still unknown). Let $x_j$ be the first item for which $p_j < 1$. The elicitor must arbitrarily choose such item for eliciting some information about $x_i$. However, it might happen that $x_j$ is the second item (besides $x_i$) in the 2-item atomic bundle. In this case, we claim that the bundle $\{x_i\}$ will never be demanded since:

$$u(x_ix_j) = 2 - (p_{x_i} + p_{x_j}) = 1 - p_{x_i} + 1 - p_{x_j} > 1 - p_{x_i} > v(x_i) - p_{x_i} = u(x_i)$$

Thus, no information about $\alpha = v(x_i)$ will be revealed.

Now, we present a class which cannot be elicited by a descending auction, but is elicitable by an ascending auction. Consider two bidders from the class described above, with an extra information that the singletons for which they have non-zero valuations are not consecutive and are not the same item. In addition, there is a third bidder with the valuation:

$$v = (x_1x_2x_3 : 2.5) \oplus (x_1x_2x_4 : 2.5) \oplus (x_1x_3x_4 : 2.5) \oplus (x_2x_3x_4 : 2.5)$$

First, we show that a descending auction cannot find the optimal allocation for every realization of the valuations. For determining the optimal allocation, we must know which bidder has the greatest value for a singleton, i.e., we must find the value of the $x_i$ for the two players with the unknown valuation. Even if we knew one of these values, we would still need to know whether the other value is smaller or greater. However, exactly the same proof as above shows that a descending auction cannot guarantee to extract any information about this unknown value.

An ascending auction can find the optimal allocation: under zero prices, both players demand the 3-item bundle with the valuation of 2.5. Thus, the singletons $x_i$ and $x_j$ with the non-zero valuations are revealed. We raise the prices of the other items (except $x_i$ and $x_j$) to $L$. Since $x_i$ and $x_j$ are not adjacent, the utility from all the bundles, except these singletons, will be negative. Raising the prices of these two items reveals the unknown $\alpha$'s. □

## B.2 Deterministic vs. Non-Deterministic Auctions

**Proof for Proposition 7:**

*Proof.* Consider bidder 1 with one of the following XOR valuations:

$$v = (ab : 3) \oplus (a : \alpha)$$

$$v = (ab : 3) \oplus (b : \beta)$$

Where $\alpha, \beta \in \{0.4, 0.6\}$ are unknown to the auctioneer. Clearly, a non-deterministic algorithm can guess the singleton, raise the other item until the singleton is demanded, and then increase the price of the singleton until the value is discovered.

No deterministic algorithm, however, can learn the valuation. For zero prices, the bidder will clearly demand the bundle $ab$. The bidder's demand could change only if the price of the bundle $ab$ is greater

[22] If the bidder's tie-breaking rule favors the grand bundle, we can raise the price of every item in turn, with a small increment, until $x_{i-1}x_ix_{i+1}$ is demanded.

than 2, i.e., when either $p_a$ or $p_b$ are greater than 1. In this case, this singleton term will not be demanded, and the valuation will not be fully elicited.

Consider bidder 1 described above, and two other bidders with the XOR valuations $v_2 = (ac : 2.5)$ and $v_3 = (bc : 2.5)$.

The optimal allocation clearly depends on the value of the singleton of bidder 1, but we saw that bidder 1's valuation cannot be learned with a deterministic ascending auction. (Note that even non-anonymous deterministic ascending auctions cannot elicit these valuations.) □

### B.3 Anonymous vs. Non-Anonymous Auctions

**Proof for Proposition 8:**

*Proof.* Consider three players with the following valuations:

$$v_1 = (ab : 2) \oplus (a : \alpha)$$

$$v_2 = (bc : 2) \oplus (b : \beta)$$

$$v_3 = (ca : 2) \oplus (c : \gamma)$$

Where $\alpha, \beta, \gamma$ are unknown values between $(0, 1)$. The optimal allocation should allocate a singleton to the bidder with the highest singleton valuation, and give the other items to the player that has a valuation of 2 for them.

A non-anonymous item-price ascending auction can easily find the optimal allocation, by raising, for each bidder, the price of the item not in his singleton (e.g., $p_b$ for bidder 1), until each bidder demands his singleton and thus revealing his unknown value.

Any anonymous auction must raise the price of some item above 1, before it encounters any change in the demands of the bidders or gaining any other information about the unknown values. No information will be elicited about the value of this item for the player that has a non-zero value for it. □

### B.4 Sequential vs. Simultaneous Auctions

**Proof for Proposition 9:**

*Proof.* Consider three bidders with the following valuations:

$$v_1 = (abc : 2) \oplus (x : \frac{1}{3})$$

$$v_2 = (abc : 2) \oplus (y : \beta)$$

$$v_3 = (a : 2) \oplus (b : 2) \oplus (c : 2)$$

Where $x, y \in \{a, b, c\}$ are unknown items, and $\beta$ is an unknown number between $(0, \frac{1}{2})$.

If $x = y$, the optimal allocation allocates $x$ to bidder 1 if $\beta < \frac{1}{3}$, or otherwise it allocates $y$ to bidder 2 (in both cases, bidder 3 receives the other items). If $x$ and $y$ are distinct, each bidder should receive one of these items, and the third item goes to bidder 3. Therefore, for determining the optimal allocation, the auctioneer has to reveal the identity of $x$, the identity of $y$ and its value.

First, we show that a simultaneous auction cannot find the optimal allocation. Bidder 1 definitely demands the whole bundle $\{abc\}$ when its price is below $\frac{3}{2}$. Since in a simultaneous auction the sum of the prices in all trajectories is equal at every stage, one of the prices for bidder 2 must exceed $\frac{1}{2}$ at this point of time. If this item turns out to be $y$, player 2 will never demand this singleton, thus the value of $\beta$ will never be revealed.

A sequential auction, however, can find the optimal allocation. We first raise bidder 1's price of some item to $\frac{5}{3}$ and thus find $x$[23]. Then, we raise the price bidder 2's price for an item different than $x$. If some singleton is demanded, we found $y$ and its value. If no singleton is demanded, it follows that $y$ and $x$ are distinct items, thus the optimal allocation do not depend on the value of $\beta$. □

[23] If no singleton is demanded, this also reveals $x$.

## B.5 Adaptive vs. Oblivious Auctions

### Proof for Proposition 10:

*Proof.* Consider the following class of XOR valuations over the three items $a, b, c$:

$$v = \{(xy : 3) \oplus (yz : 3) \oplus (x : \alpha) \oplus (z : \beta)\}$$

Where $\alpha, \beta$ are unknown values between (0,1) (may be different between bidders) and $x, y, z \in \{a, b, c\}$ are distinct items.

A simple adaptive algorithm that fully elicits this class of valuations is the following: Raise the prices of the items in the bundle demanded under zero prices by $\epsilon = \frac{\delta}{2}$ (then the bidder clearly demands the other 2-item atomic bundle). Now, raise the price of the item in the intersection of the two bundles demanded so far ('$y$'), until the bidder demands some singleton. Then raise the price of this singleton to $L$, and continue raising the price of $y$ until the other singleton is demanded. Concluding the values from the responses is then straightforward.

An oblivious algorithm cannot know in advance what is the item in the intersection of the two 2-item bundles: A necessary condition for the bidder to demand a singleton is that $p(xy) > 2$ and $p(yz) > 2$. Assume w.l.o.g. that $a$ is the first item for which the oblivious algorithm increases its price above 1. Then, for a valuation with the terms $(ab : 3) \oplus (bc : 3) \oplus (a : \alpha) \oplus (b : \beta)$ the value $\alpha$ will not be learned.

Consider a bidder 1 whose valuation is drawn from the class described above. and a bidder 2 with the XOR valuation $v = (ab : 3) \oplus (bc : 3) \oplus (ac : 3)$. The optimal allocation will allocate the singleton with the highest value to bidder 1, and the other items to bidder 2. We saw that an oblivious auction cannot learn the first valuation, but an adaptive auction can. Since the second valuation is fully known, the claim about the elicitation follows. (Note that the theorem also holds for non-anonymous auctions.) □

## B.6 Preference Elicitation vs. Full Elicitation - missing proofs

### Proof for Theorem 3:

*Proof.* Denote the $m$ goods by $a_1, ..., a_{\frac{m}{2}}$ and $b_1, ..., b_{\frac{m}{2}}$ $(m \geq 4)$. Consider the unit-demand (i.e., XOR of singletons) valuations for $1 \leq i \leq \frac{m}{2}$: $v_i = a_i \oplus b_1 \oplus b_2 \oplus ... \oplus b_{\frac{m}{2}}$, where the values of all singletons is 3, except for $b_i$ which is only known to be in $(0, 1)$. Denote this low value for $b_i$ by $\underline{b_i}$.[24] Now let $v = v_1 \vee v_2 \vee ... \vee v_{\frac{m}{2}}$ (see definition of the $\vee$ operator in Appendix D.2), and each $v_i$ will be called a term. This valuation is an OXS valuation by definition (see definition in Appendix D.2), and thus it holds the substitutes property ([16]).

First, we show that for every $1 \leq i \leq \frac{m}{2}$, the only bundle where the value $\underline{b_i}$ has an effect on the value of the bundle, is $b_i, a_1, ..., a_{i-1}, a_{i+1}, ..., a_m$ (denote this bundle as $\overline{a_{-i}b_i}$). Let $B$ be some bundle in which the item $b_i$ contributes $\underline{b_i}$ to the value. All the XOR terms must be used for calculating the value, otherwise $b_i$ could have contributed 3 in the unused term. If there is a set of terms, except the $i$th term, in which the value is calculated according to the values of the $b$'s, then a permutation of the $b$'s among the terms must achieve a contribution of 3 for any of them. Thus, $|B| = \frac{m}{2}$, and it contains $b_i$ and all $a_j$ for $j \neq i$.

Thus, in order to learn the value $\underline{b_i}$, bidder $i$ must demand the bundle $a_{-i}b_i$ at least once along the ascending trajectory. Let $\overrightarrow{p} = (p_1, ..., p_m)$ be the vector of prices for which $a_{-i}b_i$ is demanded. Thus, the utility from this bundle is not smaller than the utility from the bundle $a_{-i-j}b_i$ (i.e., when we remove some item $j \neq i$ from $a_{-i}b_i$). Therefore, $\underline{b_i} + 3 - p_{b_i} - p_{a_j} \geq 3 - p_{b_i}$ (recall that the prices are linear). We conclude that $p_{a_j} \leq \underline{b_i} < 1$. $a_{-i}b_i$ will also be preferred over the bundle $a_{-i}a_i$ (i.e., when we replace $b_i$ with $a_i$). Thus, $\underline{b_i} - p_{b_i} \geq 3 - p_{a_i}$ and we get $p_{a_i} \geq 3 - \underline{b_i} + p_{b_i} \geq 3 - \underline{b_i} > 2$. We see that when $a_{-i}b_i$ is demanded, $p_{a_i} > 2$ and $p_{a_j} < 1$ for any $j \neq i$.

[24] For example, the following 4-item valuation cannot be fully elicited by a single item-price auction ($\underline{b_1}, \underline{b_2}$ are unknown values in $(0, 1)$):

$$v = (\, (a_1 : 3) \oplus (b_1 : \underline{b_1}) \oplus (b_2 : 3) \,) \vee (\, (a_2 : 3) \oplus (b_1 : 3) \oplus (b_2 : \underline{b_2}) \,)$$

Now, let $\overrightarrow{q} = (q_1, .., q_m)$ be a price vector for which the bundle $a_{-j}b_j$ is demanded (for some $j \neq i$). >From symmetry, we must similarly have that $p_{a_j} > 2$ and $p_{a_i} < 1$. Therefore, $p$ and $q$ cannot be on the same ascending price trajectory, and only one of them could be demanded in an ascending auction. Since $i, j$ were chosen arbitrarily, it follows that in a single ascending trajectory we can only discover the value of one of the $b_i$'s. Thus, we need at least $\frac{m}{2}$ ascending trajectories in order to learn $v$. □

### B.7 OXS Valuations Have Compact representation

**Lemma 1.** *An OXS valuation has at most $m$ XOR terms. Thus, any OXS valuation can be represented by no more than $m^2$ values.*

*Proof.* We prove that for any bundle $S$ and any item $a \in S$, calculating $v(S \setminus a)$ uses a subset of the terms needed for calculating $v(S)$. Since clearly only $n$ terms are used for valuating $M$, it follows (inductively) that any bundle will be calculated according to a subset of these terms, and the lemma follows.

Consider some bundle of $k$ items $S = (a_1, ..., a_k)$, and let $t_1, ..., t_k$ be the terms in which the items are valuated, respectively. When valuating the bundle $S \setminus a_1$, if no item is valuated in the term $t_1$, then the other items $a_2, ..., a_k$ are valuated in exactly the same terms as when valuating $S$ (otherwise the value for $S$ could increase). If there is an item in $S \setminus a_1$ that is valuated in $t_1$, w.l.o.g $a_2$, then we check if the term $t_2$ is used by the items $a_3, ..., a_k$, and we proceed (by induction), until a stage $j$ where the term $t_j$ is left unused (and then $a_{j+1}, ..., a_k$ are valuated in their "old" terms), or until the last item $a_k$ is valuated in the term $t_{k-1}$. (We assume w.l.o.g. that the items are indexed in the order derived by the proof.) Therefore, in any case, an item will not take its value from a term not in $\{t_1, ..., t_k\}$. □

## C Bundle-price Ascending Auctions - Missing Proofs

### C.1 NBEA Auctions

**Proposition 12.** *NBEA auctions find the optimal allocation for $k$-term XOR valuations within a number of steps which is polynomial in $k$.*

*Proof.* For this proof, we assume that bidders report the smallest bundle in their demand sets.[25] NBEA auctions increase, in each step, the price of some bundle in the demand set by $\epsilon$. Since bidders demand only bundles with minimal number of items, clearly only bundles which are terms (atomic bids) in the XOR formula will be demanded. In the worst case, the auction terminates when the price of all bundles equals their value. Therefore, the total number of steps is at most $k \cdot L \cdot n \cdot \frac{1}{\epsilon}$. □

**Proof for Proposition 11**:

*Proof.* Consider a bidder with an additive valuation that assigns the values $1, 2, ..., m$ to the singletons $1, ..., m$ respectively, and a second bidder that assigns these values to the items in reversed order, i.e., the values $1, 2..., m$ to the singletons $1, ..., m$ respectively. (Assume, for simplicity, that $m$ is even.) The optimal allocation must allocate the items $\{\frac{m}{2} + 1, ..., m\}$ to the first bidder and the other items $(\{1, ..., \frac{m}{2}\})$ to the second bidder. Note that there are at least $2^{\frac{m}{2}} - 1$ different bundles with strictly greater values than $v_1(\{\frac{m}{2}+1, ..., m\})$ (one for any subset of $\{1, ..., \frac{m}{2}\}$). Since the auction is conservative, the bidder can only demand a single bundle in each step, and he can only be assigned a bundle he has requested. Thus, the bidder will demand an exponential number of different bundles before the auction ends.[26]

If the bidders submit all the bundles in their demand set at every stage, and if the highest valuation is relatively small, than conservative auction terminate within a polynomial number of steps. Under zero prices, the maximal surplus of every bidder is $L$. Conservative auction can raise, in each step, the

[25] Due to the definition of XOR valuations, they do not have any incentive to ask for a bigger bundle.

[26] Note that the utility from any of the other bundles will be greater than the utility from $\{\frac{m}{2} + 1, ..., m\}$), until its price is raised (at least once).

price of all the bundles some bidder demanded so far by $\delta$. Then, the maximal surplus of some bidder decreases by $\delta$ in any stage. Since the auction terminates when the maximal surplus of all the bidders is zero, than the number of steps in the auction will clearly be smaller than $n \cdot \frac{L}{\delta}$, and recall that $L$ is polynomial in $m, n$ and $\frac{1}{\delta}$. □

## C.2 Limitations of Anonymous Bundle-Price Ascending Auctions

### Proof for Theorem 5:

*Proof.* Assume we have $n$ bidders and $n^2$ items for sale, and that $n$ is prime. We construct $n^2$ distinct bundles with the following properties: for each bidder, we define a partition $S^i = (S_1^i, ..., S_n^i)$ of the $n^2$ items to $n$ bundles, such that any two bundles from different partitions intersect. In Appendix C.3 below we show an explicit construction using the properties of linear functions over finite fields. The rest of the proof is independent of the specific construction.

Using these $n^2$ bundles we construct a "hard-to-elicit" class. Every bidder has an atomic bid, in his XOR valuation, for each of these $n^2$ bundles. A bidder $i$ has a value of 2 for any bundle $S_j^i$ in his partition. For all bundles in the other partitions, he has a value of either 0 or of $1-\delta$, and these values are unknown to the auctioneer. Since every pair of bundles from different partitions intersect, only one bidder can receive a bundle with a value of 2.

No bidder will demand a low-valued bundle, as long as the price of one of his high-valued bundles is below 1 (and thus gain him a utility greater than 1). Therefore, for eliciting any information about the low-valued bundles, the auctioneer should first arbitrarily choose a bidder (w.l.o.g bidder 1) and raise the prices of *all* the bundles $(S_1^1, ..., S_n^1)$ to be greater than 1. Since the prices cannot decrease, the other bidders will clearly never demand these bundles in future stages. It might happen that the low values of all the bidders for the bundles not in bidder 1's partition are zero (i.e., $v_i(S_j^1) = 0$ for every $i \neq 1$ and every $j$), however, allocating each bidder a different bundle from bidder 1's partition, might achieve a welfare of $n+1-(n-1)\delta$ (bidder 1's valuation is 2, and $1-\delta$ for all other bidders); If these bundles were wrongly allocated, only a welfare of 2 might be achieved (2 for bidder 1's high-valued bundle, 0 for all other bidders). We conclude that anonymous bundle-price auctions cannot guarantee a welfare greater than 2 for this class, where the optimal welfare can be arbitrarily close to $n+1$. □

## C.3 Explicit construction for a combinatorial design

Given $n$ bidders and $n^2$ items ($n$ is prime), we give below an explicit construction for $n^2$ distinct bundles, composed of $n$ different $n$-item partitions in which bundles from different partitions intersect. This combinatorial design is used in the proof for Theorem 5.

*Proof.* We need to construct $n^2$ bundles, each of size $n$ with the following properties: for each bidder, we define a partition $S^i = (S_1^i, ..., S_n^i)$ of the $n^2$ items to $n$ bundles of size $n$, such that any two bundles from different partitions intersect (i.e., $S_j^i \cup S_l^k \neq \emptyset$ for every $i \neq k$ and every $l, j$).

We show an explicit construction of such bundles using the properties of linear functions over finite fields (for that, we denote the bidders by $0, ..., n-1$):

Recall that $Z_n = \{0, ..., n-1\}$ is a field if (and only if) $n$ is prime. Denote the $n^2$ items for sale by pairs of numbers in $Z_n$. Each linear function $ax+b$ over the finite field $Z_n$ denotes an $n$-item bundle (a total of $n^2$ bundles where $a, b \in Z_n$). The items in each bundle are the pairs $(x, ax+b)$ for every $x \in Z_n$. The bundles assigned to bidder $i$ are the $n$ bundles $ix+b$ where $b \in Z_n$. We need to show that the bundles assigned to bidder $i$ form a partition, and indeed the functions $ix+b_1$ and $ix+b_2$ cannot intersect when $b_1 \neq b_2$. It is also easy to see that every two bundles that are assigned to different bidders do intersect: consider the functions $ix+b_1$ and $jx+b_2$. Since $z_n$ is a field, clearly an $x$ exist such that $x(j-i) = (b_1-b_2)$ when $j \neq i$ for any $b_1, b_2$. The $j$th bundle of bidder $i$ is therefore, $S_j^i = \{(0, i \cdot 0 + j), ..., (n-1, i \cdot (n-1) + j)\}$. □

**A descending auction for bidders with submodular valuations:**

**Initialization:** set all item prices to $L$. Let $X_i$ be the current items allocated to bidder $i$, and for each bidder initialize $X_i \leftarrow \emptyset$.

**Repeat:** For all items $i = 1, ..., m$ (the items are arbitrarily ordered), decrease the price $p_i$ of item $i$ by $\epsilon = \delta$.

Allocate the item to the first bidder $j$ that demand his current bundle $X_j$ together with item $i$ (i.e., $X_j \leftarrow X_j \cup \{i\}$).

Figure 5: This item-price descending auction guarantees at least $\frac{1}{2}$ of the optimal welfare for submodular valuations. We do not know if there is an ascending auction achieving the same approximation ratio.

# D Missing Proofs and Definitions

## D.1 A Descending Auction for Sub-Modular Valuations

**Proposition 13.** *For any profile of sub-modular valuations, the descending auction described in Figure 5 achieves at least $\frac{1}{2}$ of the social welfare.*

*Proof.* The algorithm by Lehmann, Lehmann and Nisan arbitrarily orders the items and allocates each item in turn to the bidder with the highest marginal valuation for it (given the items already allocated to him). The descending auction decreases the price of each item $i$, until a bidder demands it together with the bundle $S$ he already owns. At this stage, up to the $\epsilon$ used, his utility from the bundle $S \cup \{i\}$ is zero, thus his value for this bundle equals its current price, i.e. $v(S \cup \{i\}) = \sum_{j \in S \cup \{i\}} p_j$. Similarly, this bidder valuates $S$ by $v(S) = \sum_{j \in S} p_j$. Thus, $p_i = v(S \cup \{i\}) - v(S)$ is, by definition, the marginal valuation of this bidder for item $i$. By decreasing the price of item $i$, we exactly found the bidder with the highest marginal valuation for it. □

## D.2 Classes of Valuations

**Definition 2.** ([19]) Let $v_1, v_2$ be two valuations. We define the valuation $v_1 \vee v_2$ by:

$$(v_1 \vee v_2)(S) = \max_{T \subseteq S} (v_1(T) + v_2(S \setminus T))$$

**Definition 3.** A valuation $v$ is an *OXS valuation* if it can be represented as $v = v_1 \vee v_2 \vee ... \vee v_k$ where each $v_i$ is a XOR of singletons (i.e., a XOR valuation where the size of each term is 1).

**Definition 4.** [14] A valuation $v$ is said to satisfy the *substitutes (or gross-substitutes)* property if for any price vectors $\overrightarrow{q} \geq \overrightarrow{p}$ (coordinatewise comparison), if $S = \{j \in M | p_j = q_j\}$ and $A$ maximizes the bidder's utility under the price vector $\overrightarrow{p}$, then there exists a bundle $B$ that maximizes the bidder's utility under the price vector $\overrightarrow{q}$ such that $S \cap A \subseteq B$.

# E Simulating Queries by Demand Queries

In a companion paper [7], we showed the the following queries can be simulated by a polynomial number of demand queries (polynomial in $n, m$ and $\log L$). Some of these queries can be found in the literature, and we found the others natural or useful. Here, we show that these queries can be simulated by a pseudo-polynomial number of *ascending* demand queries (polynomial in $n, m$ and $L/\delta$).

1. *Value query*: The auctioneer presents a bundle $S$, the bidder reports his value $v(S)$ for this bundle.

2. *Marginal-value query*: The auctioneer presents a bundle $A$ and an item $j$, the bidder reports how much he is willing to pay for $j$, given that he already owns $A$, i.e., $v(j|A) = v(A \cup \{j\}) - v(A)$.

3. *Demand query (with item prices)*: The auctioneer presents a vector of item prices $p_1 ... p_m$; the bidder reports his demand under these prices, i.e., some set $S$ that maximizes $v(S) - \sum_{i \in S} p_i$.

4. *Indirect-utility query*: The auctioneer presents a set of item prices $p_1...p_m$, and the bidder responds with his "indirect-utility" under these prices, that is, the highest utility he can achieve from a bundle under these prices: $max_{S\subseteq M}(v(S) - \sum_{i\in S} p_i)$.

5. *Relative-demand query*: The auctioneer presents a set of non-zero prices $p_1...p_m$, and the bidder reports the bundle that maximizes his value per unit of money, i.e., some set that maximizes $\frac{v(S)}{\sum_{i\in S} p_i}$.

Due to Proposition 1, a value query can be simulated by an ascending auction.

**Proposition 14.** *Marginal-value queries can be simulated by an ascending item-price auction. This simulation requires a polynomial number of queries.*

*Proof.* Simulating $v(j|S)$: start with zero prices, and increase the prices of all the items in $M \setminus \{S \cup \{j\}\}$ to be $L$. Then, gradually increase $p_j$ by $\delta$ and stop when the bidder stops demanding $M \setminus \{S \cup \{j\}\}$- and this price for the item $j$ is $v(j|S)$. (A $\delta$ may be added to this value, as derived from the tie breaking rules of the bidder). □

**Proposition 15.** *Indirect-utility queries can be simulated by an ascending item-price auction. This simulation requires a polynomial number of queries.*

*Proof.* For simulating $IU(\overrightarrow{p})$ using demand queries, we first ask the bidder for his desired bundle under these prices $S = D_i(\overrightarrow{p})$. Then, we calculate $v(S)$ according to the procedure described in Propositions 1 and 2. □

**Proposition 16.** *Relative-demand queries can be simulated by an ascending item-price auction. This simulation requires a polynomial number of queries.*

*Proof.* We simulate $RD(\overrightarrow{p})$ by the following ascending auction:
*Initialization: start with a price vector $\epsilon\overrightarrow{p}$ ($\epsilon > 0$).*
*Repeat: for the vector of prices $\overrightarrow{q}$, if the bidder demands a non empty set, raise prices to $\overrightarrow{q} + \epsilon\overrightarrow{p}$.*
*Finally: If the bidder demands the empty set at stage $t+1$, terminate the auction, and return the bundle $S$ demanded at stage $t$ as the answer.*

Now we show that for the price vector $\overrightarrow{p}$, every other bundle $T$ has a smaller relative weight than $S$ (up to $\epsilon$), i.e.,

$$\frac{v(S)}{p(S)} \geq \frac{v(T)}{p(T)} - \epsilon. \tag{E.1}$$

At time $t$, the bundle $S$ was demanded, therefore $v(S) - \epsilon tp(S) \geq 0$. Thus, $\frac{v(S)}{p(S)} \geq \epsilon t$. Assume that inequality E.1 does not hold, then it follows that $\frac{v(T)}{p(T)} - \epsilon > \epsilon t$, or $v(T) > \epsilon(t+1)p(T)$. But in time $t+1$ no bundle achieved a positive utility since the empty set was demanded. Contradiction. □

# F Efficiency of Existing protocols

Consider the following definition for competitive equilibrium with non-anonymous bundle prices:

**Definition 5.** Non-anonymous bundle prices $\overrightarrow{p}$ and an allocation $X = X_1, ..., X_n$ are in *competitive equilibrium* if:

- The bundle allocated for each bidder maximizes his utility under the current prices, i.e., for any other bundle $Y_i \subseteq M$, $v_i(X_i) - p_i(X_i) \geq v_i(Y_i) - p_i(Y_i)$.
- The allocation $X$ maximizes the *seller's revenue* under the current prices, i.e., for any other allocation $Y = Y_1, .., Y_n$, $\sum_{i=1}^{n} p_i(X_i) \geq \sum_{i=1}^{n} p_i(X_i)$.

*Claim* 2. In any competitive equilibrium $\overrightarrow{P}, X$ the allocation is efficient.[27]

[27]Many proofs exist for equivalent statements, see, e.g., the recent survey by Parkes [22].

**Item-price ascending auction for substitute valuations:**
**Initialization:** For every item $j \in M$, set $p_j \leftarrow 0$.
For every bidder $i$ let $S_i$ be his current bundle, initialized to $\emptyset$.
(To *any* price vector in this auction, we coordinatewise add the price vector $\{\frac{\delta}{m \cdot 2^{-i}}\}_{i \in M}$ for preventing indifference between bundles.)
**Repeat:** For every bidder $i$ and given the current price vector $\overrightarrow{p}$, let $\overrightarrow{q^i}$ be the price vector where $q^i_j = p_j$ if $j \in S_i$, and $q^i_j = p_j + \epsilon$ otherwise.
Let $D_i$ denote the bundle demanded by bidder $i$ at the prices vector $\overrightarrow{q^i}$.
For some bidder $i$ such that $S_i \neq D_i$ update:
- For every item $j \in D_i \setminus S_i$, $p_j \leftarrow p_j + \epsilon$ (where $\epsilon = \frac{\delta}{m \cdot n}$)
- $S_i \leftarrow D_i$
- For every bidder $k \neq i$, $S_k \leftarrow S_k \setminus D_i$

**Finally:** Run until you reach $\overrightarrow{p}$ where for every bidder $i$, $S_i = D_i$.

Figure 6: This version of the item-price ascending auction by [14, 9, 11] ends up with the optimal allocation where the bidders' valuations holds the gross-substitute property.

*Proof.* Let $\overrightarrow{P}, X$ be a competitive equilibrium, and consider some allocation $Y = (Y_1, ..., Y_n)$. Since each bidder $i$ maximizes his utility, $v_i(X_i) - p_i(X_i) \geq v_i(Y_i) - p_i(Y_i)$. By summing over all the bidders, and rearranging the summands we get:

$$\sum_{i=1}^{n} v_i(X_i) \geq \sum_{i=1}^{n} v_i(Y_i) + \sum_{i=1}^{n} p_i(X_i) - \sum_{i=1}^{n} p_i(Y_i)$$

Since $\sum_{i=1}^{n} p_i(X_i) \geq \sum_{i=1}^{n} p_i(Y_i)$, the allocation $X$ achieves a better welfare than $Y$:

$$\sum_{i=1}^{n} v_i(X_i) \geq \sum_{i=1}^{n} v_i(Y_i)$$

□

**Proposition 17.** *NBEA auctions (see Figure 4) always end up with the optimal allocation.*

*Proof.* The termination condition for NBEA auctions ensures that this auction ends up in a competitive equilibrium as described above in Definition 5. The only subtle issue is that the allocation maximizes the utility of the bidders up to $\epsilon$ (since we update one price at a single stage). Due Claim 2 above, the allocation is therefore efficient up to an additive term of $n \cdot \epsilon$. Choosing, e.g., $\epsilon < \frac{\delta}{n}$ ensures that the allocation will be efficient up to $\delta$. □

**Theorem 6.** *The auction described in Figure 6 finds the optimal allocation (up to $\delta$) if all bidders have substitutes valuations.*

*Proof.* We first claim that due to the definition of substitutes valuations, once an item is demanded, it will be demanded by some bidder until the end of the auction. Clearly, items which are not demanded by the bidders at any stage of the auction will end up with zero prices.

For that, we distinguish between three types of bidders at some stage $t$ of the auction: (1) the bidder $i$ that receives new items to $S_i$ (2) bidders that items are subtracted from their previous bundles (3) none of the above. >From the definition of the individual price vectors (the $\overrightarrow{q^i}$'s), bidders of type (1) and (3) will clearly be presented with the exactly the same price vector – a price $p_j$ for the items in their (old) bundle, and a price of $p_j + \epsilon$ for any other item $j$ – thus their demand will not change. Given a bidder $l$ of type (2) who demanded $S_l$ in the previous round, the prices of the items in $S_i \setminus S_l$ were increased now by $\epsilon$. Note that these items are now in the new $S_i$. What is left to be shown is that the items in $S_l \setminus S_i$ will still be demanded by bidder $i$: this follows from the definition of substitutes valuations, where the bidder must demand the remaining items in some bundle from his demand set.

Since we added to the prices exponentially-small perturbations (see Figure 6), we know that no ties exist, thus the bidder will surely demand these items. We conclude that all the items demanded in round $t$ will be demanded by some bidder in round $t+1$, and thus at the final stage.

We now show that the final allocation is optimal (up to $\delta$). The auction terminates when exactly $S_i$ is demanded by each bidder. Since bidders respond to slightly different price vectors ($\epsilon$ is added to the items they do not hold), it may happen that they would prefer other bundles under the price vector $\overrightarrow{p}$ (but by no more than $|M \setminus S_i| \cdot \epsilon < m\epsilon$). Thus, we do know that for every bidder $i$ and every bundle $T_i$:

$$v_i(S_i) - p(S_i) \geq v_i(T_i) - p(T_i) - m\epsilon$$

Therefore, for any partition $T_1, ..., T_n$, summing over all bidders we get:

$$\sum_{i=1}^{n} v_i(S_i) - \sum_{i=1}^{n} p(S_i) \geq \sum_{i=1}^{n} v_i(T_i) - \sum_{i=1}^{n} p(T_i) - nm\epsilon$$

Since unallocated items have zero prices, $\sum_{i=1}^{n} p(S_i) = \sum_{i=1}^{n} p(T_i)$. This, with setting $\delta = nm\epsilon$ derives that:

$$\sum_{i=1}^{n} v_i(S_i) \geq \sum_{i=1}^{n} v_i(T_i) - \delta$$

Thus, the auction finds an optimal allocation up to $\delta$. □