Multilateral Deferred-Acceptance Mechanisms

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Abstract. We study the design of multilateral markets, where agents with several different roles engage in trade. We first observe that the modular approach proposed by Dütting et al. [6] for bilateral markets can also be applied in multilateral markets. This gives a general method to design Deferred Acceptance mechanisms in such settings; these mechanisms, defined by Milgrom and Segal [13], are known to satisfy some highly desired properties.
We then show applications of this framework in the context of *supply chains*. We show how existing mechanisms can be implemented as multilateral Deferred Acceptance mechanisms, and thus exhibit nice practical properties (as group strategy-proofness and equivalence to clock auctions). We use the general framework to design a novel mechanism that improves upon previous mechanisms in terms of social welfare. Our mechanism manages to avoid "trade reduction" in some scenarios, while maintaining the incentive and budget-balance properties.

1 Introduction

Markets are often characterized by multiplicity of participants with diverse characteristics. While much of the mechanism-design literature focuses on bilateral settings that only distinguish between two types of agents - buyers and sellers, many real life applications require a more complex description of markets. For instance, dealing with markets in which buyers wish to purchase a bundle of items that are sold separately by different sellers might require a distinction between different types of sellers. Further and more complex distinctions will be needed as markets become more complex and involve trade between diversified agents.

In settings of bilateral trade, and nonetheless in more complex settings where agents engage in several bilateral transactions, it is generally impossible to achieve an efficient allocation while maintaining agents' participation constraints, incentive compatibility and budget balance [15]. Therefore, it is needed to sacrifice one of these goals in order to fully achieve the others. For instance the VCG mechanism maintains agents' incentives and implements the efficient allocation but is generally not budget balanced. Other mechanisms relax incentive compatibility to achieve the efficient allocation and budget balance (see [10] for a survey).

In this paper we devise a family of mechanisms named multilateral deferred-acceptance (MDA) mechanisms. These mechanisms apply the methodology of deferred-acceptance (DA) auctions, introduced by Milgrom and Segal [13], to multilateral markets.[3] DA auctions set allocations using an iterative process of rejecting the least attractive bid according to a carefully defined ranking function. Combining this sort of algorithm with threshold payments yields a mechanism with strong incentive properties - other than being truthful, the DA auction is also weakly group-strategy proof (WGSP). This means that no coalition of agents has a joint deviation from truthful bidding that is strictly profitable for all members of the coalition. Another desired feature of DA auctions is that they are equivalent to clock auctions, an auction format which is intuitive for bidders and is thus considered practical. WGSP and equivalence to clock auctions are desired properties of DA auctions that are not generally attained by other mechanisms. For example, the VCG mechanism and greedy mechanisms such as [9] do not possess these properties.

Dütting et al. [6] took a modular approach to adapt DA auctions to two-sided markets. In this paper, we take the general concept introduced in [6] one step forward and observe that their modular approach can also

[3] Milgrom and Segal's study was motivated by the US government's attempt to reallocate parts of the radio spectrum currently allocated for television broadcasting to mobile internet use. The reallocation problem involves purchasing broadcast rights from TV stations and reselling cleared spectrum parts to mobile phone companies. Other than computational difficulties, this problem raises several design issues. The most prominent ones are maintaining agents' incentives, keeping procurement costs low relative to expected proceeds and designing a simple and intuitive auction that will make truthful bidding an obvious dominant strategy for agents.

be applied to multilateral markets with several types of agents. Similar to [6], the mechanism's operation is determined by two elements: separate rankings for each set of agents and a composition rule. In each period the composition rule selects few classes (or groups) of agents. Each class selected will be examined and the least desirable agent of that class, according to the corresponding ranking, will be rejected.[4] When the mechanism terminates, all unrejected agents are declared winners and threshold payments are set. We generalize the result by [6] and show that any mechanism from this family (MDA mechanisms) is equivalent to a one-sided DA auction; thus, it is strategy-proof, individually rational, WGSP and equivalent to a clock auction.

After introducing the class of MDA mechanisms, we apply them in the context of *supply chains* (see, e.g.,[20, 7, 21, 1, 19]). Supply chains are collections of markets where each agent engages in at least one bilateral trade, either as a seller of an item he produces, or as a buyer of items (that are used for consumption or as inputs for production). Supply chains are thus composed of several two-sided markets and feature the same impossibilities that exist in bilateral-trade settings of maximizing social welfare while maintaining agents' incentives and budget balance.

We study a model for supply chains that was introduced by Babaioff and Nisan [2] and Babaioff and Walsh [3]. In this model, the supply chain can be viewed as a directed tree-graph with a node per each good. Ingoing edges to a node define the inputs for the production of the relevant good, and producers incur a manufacturing cost which is private information. [2] and [3] showed a dominant-strategy truthful, budget-balanced mechanism that waives only the least profitable trade (or *"procurement set"*, which is a minimal trade cycle in a supply chain and typically involves multiple agents).

Our first result for supply chains shows that the trade reduction mechanisms of [2] and [3] can be implemented as MDA mechanisms. Thus, other than being IR, strategy-proof and budget balanced (as proven in [2] and [3]), these trade reduction mechanisms are also WGSP and equivalent to clock auctions. This is shown under the assumption of homogeneous demand (i.e., all end consumers demand the same bundle).[5]

Our second and main result shows how to use the machinery of MDA mechanisms to construct a novel mechanism that provides an improved outcome in terms of social welfare compared to the above trade reduction mechanisms. This mechanism is not only strategy-proof and budget balanced, but also WGSP and equivalent to clock auctions. It operates by iteratively rejecting procurement sets, but unlike the trade reduction mechanism that always waives one procurement set that engages in trade in an efficient outcome, our mechanism will sometimes result in the efficient allocation (with no reduction of valuable trades). In settings where the efficient allocation is composed of numerous distinct procurement sets, the improvement in terms of social welfare may admittedly be less significant; however, in markets where the efficient allocation consists of a small number of procurement sets, this improvement may be substantial. We ran some computer simulations of a simple supply chain network (see details in main body of the paper). In the simulations, our new mechanism improves upon the trade reduction mechanisms in around 17% of the instances and saves up to 100% of the overall efficiency. This provides a good indication that the improvement in efficiency is not a rare phenomenon and can be significant.

Following the rules of MDA mechanisms, our mechanism decides whether to omit one procurement set or not after observing the bids of agents that have already been declared losers. Using their values, one can bound the payments of the active agents and therefore identify situations where the efficient outcome can be implemented while maintaining a balanced budget.[6]

While designing mechanisms for more general models of supply chains is challenging, we provide a step in this direction in our final result. In this result, we relax the assumption of unique manufacturing technology. We show how an MDA mechanism can be constructed for scenarios where a certain good can be produced by different technologies that may involve different types of inputs. We show that the efficiency loss in our mechanism is limited, but we allow a trade reduction of a procurement set of each type.

[4] [6] define a mechanism for two-sided markets which iteratively rejects buyer-seller pairs. The composition rule in this mechanism determines whether rejections continue or the algorithm terminates. We allow for different compositions of agents to be rejected in each period so our composition rule serves an additional purpose of indicating the types of agents to be rejected.

[5] Our model is a generalization of the linear model in [2]; [3] did not require homogeneous demand.

[6] We note that the straightforward attempt to decide whether trade reduction should be made (namely, run VCG if it is budget balanced, or have a trade reduction otherwise) is unfortunately not truthful.

The seminal paper by McAfee [11] introduced the trade reduction technique for constructing budget-balanced mechanisms that are nearly efficient.[7] McAfee's mechanism was given for two-sided markets with unit-demand buyers and unit-supply sellers. This mechanism either implements the efficient allocation or reduces the least valuable profitable trade. [6] proved that the trade reduction mechanism for two-sided markets (a simplified version of [11] that always eliminates one valuable trade) can be implemented via a DA mechanism and it is therefore WGSP and equivalent to a clock auction. Our work is inspired by [11] in several ways. First, we sacrifice efficiency in order to satisfy incentive constraints and budget balance and our mechanism loses at most the least valuable procurement set. In addition, McAfee's mechanism computed a price as a function of the "best" losing bids, and if this price cleared the market, no trade reduction would take place. Our mechanism acts in the same spirit and sometimes implements the efficient allocation, but it is not a generalization of McAfee's mechanism. In fact, our mechanism always omits one trade when applied to the degenerate supply chain that consists of a single two-sided market with unit-demand buyers; the benefits of our mechanism stem in more complex markets.

The paper is organized as follows: Section 2 describes the general family of deferred acceptance mechanisms. Section 3 formally defines MDA mechanisms and presents their main properties. Section 4 introduces the applications discussed above of MDA mechanisms to supply chains.

2 General Deferred-Acceptance Auctions

Consider a set N of single-parameter agents and let $\mathcal{F} \subseteq 2^N$ be the set of feasible sets of agents. An allocation in this setting is represented by a set of winning agents $A \in \mathcal{F}$. Every agent $i \in N$ is characterized by a type t_i such that given an allocation A and payments $\{p_i\}_{i\in N} \subseteq \mathbb{R}$, agent i's utility is $t_i + p_i$ if $i \in A$ and p_i if $i \notin A$. An agent's type is assumed to be private information.

In this setting, [13] define DA auctions. Each agent $i \in N$ is required to submit a single bid from a finite set of possible bids $B_i \subseteq \mathbb{R}$. According to submitted bids, an iterative process of rejecting agents is preformed. All agents that are not rejected in the process are declared winners and receive the threshold payments. We now describe this process in detail.

An agent is considered **active** in iteration t if he has not been rejected in any iteration prior to t. Let $A_t \subseteq N$ denote the set of active agents at the beginning of iteration t. Each active agent is assigned a score which is a function of his bid and the bids of all previously rejected agents:

Definition 1. *[13] A* ***DA scoring function*** *is a function of the form* $\sigma_i^t : B_i \times B_{N\setminus A_t} \to \mathbb{R}_+$ *that is non-decreasing in the first argument.*

The scoring functions form a ranking over the set of active agents in which higher ranked agents are considered less attractive. Following this logic, the DA algorithm iteratively rejects the agents with the highest score, until all agents have a score of zero. Formally:

Definition 2. *[13] Given DA scoring functions, a* ***DA algorithm*** *is defined as follows: All bidders are initially active. If all active bidders have a score of zero, the algorithm terminates and the remaining active bidders are declared winners. Otherwise, the algorithm rejects the active bidders with the highest score, removing them from the active set, and iterates.*

A DA auction can now be formally defined as follows:

Definition 3. *[13] A* ***DA auction*** *is a sealed-bid auction which computes an allocation using a DA algorithm and makes the corresponding threshold payments to winners.*[8] *Losing agents are paid zero.*

DA auctions have some unique economic properties, including weak group strategy-proofness and equivalence to clock auctions. We now formally define these two concepts, using similar definitions to the ones used in [13].

[7] There is a vast literature on the efficiency of two-sided auctions that followed [11], see, e.g., [16, 17, 4, 8]. A study of the performance of DA auctions in terms of social welfare appeared in [5].

[8] Threshold payments will be formally defined in the next section. Informally, these are the highest bids for a winning agent such that he remains a winner.

Definition 4. *A mechanism is **weakly group strategy-proof (WGSP)** if for every profile of truthful reports b, every set of agents $S \subseteq N$ and every strategy profile b'_S of these agents, at least one agent in S has a weakly higher payoff from the profile of truthful reports b than from the strategy profile $(b_{N \setminus S}, b'_S)$.*

In other words, a mechanism is WGSP if no coalition of agents can do strictly better by misreporting their values, given that all other agents report their true values.

Definition 5. *A descending **clock auction** is a dynamic mechanism that presents a descending sequence of prices to each bidder. Each presentation is followed by a decision period in which each bidder decides whether to exit or continue. When the auction ends, the bidders that have never exited are declared winners and are paid their last (lowest) accepted prices.*

Theorem 6. *[13] Any DA auction is individually rational (IR), strategy-proof, WGSP and equivalent to a clock auction.*

Milgrom and Segal [13] show that several previously known mechanisms (e.g., [14], [12], [18]) can be implemented as DA auctions and thus inherit all the properties specified in Theorem 6. In section 4.3 we take a similar approach by showing that the mechanisms in [2] and [3] can be implemented as MDA mechanisms and thus inherit their properties.

3 Multilateral Deferred-Acceptance Mechanisms

We now turn to settings of multilateral markets in which agents might differ in several aspects other than their types. Consider a setting in which agents have some distinct and known characteristics which allow sorting them into different classes. Let K be the number of agents' classes and denote by N_k the set of agents of class $k \in \{1, ..., K\}$. Thus, the set of all agents N can be described as a union of K disjoint sets $N = N_1 \cup N_2 \cup \cdots \cup N_K$.[9]

We now define a family of mechanisms which we refer to as multilateral deferred-acceptance (MDA) mechanisms. First we define the MDA algorithm which sets the allocation. Later, the definition of the mechanism is completed with the description of monetary transfers.

3.1 Multilateral Deferred-Acceptance Algorithms

For all $i \in N$ let $B_i \subseteq \mathbb{R}$ be the finite set of possible bids for bidder i, so the input for the MDA algorithm is a vector $b \in \Pi_{i \in N} B_i$.[10] In the spirit of [6], the MDA algorithm is composed of two elements: **scoring functions** and **composition functions**. We now define these two elements and describe how they construct an MDA algorithm.

Scoring Functions are defined similarly to [13] (Definition 1):

Definition 7. *For each $k = 1, ..., K$ and $i \in N_k \cap A_t$, agent i's **scoring function** $s^t_{k,i} : B_i \times B_{N_k \setminus A_t} \to \mathbb{R}_+$ is non-decreasing in the first argument and assures no ties between agents of the same class.*[11]

[9] In a two-sided market with producers and consumers of a homogeneous good, N_1 might be the set of producers and N_2 might be the set of consumers. In that case producers' types will be thought of as production costs, so a producer with a cost c_i will have utility of $-c_i + p_i$ if $i \in A$, and p_i otherwise. Consumers' types will be thought of as their value from possessing one item of the traded good.

[10] As mentioned in Section 2, [13] define DA auctions with finite bid spaces. In order to use their results, we do the same. This also requires a more delicate definition of truthfulness and we follow the definition of strategy-proofness in [13] which uses the standard dominant-strategy truthfulness, only with taking care of the finite bid space. We refer the readers to [13] for the exact definition.

[11] The scoring functions are denoted with superscript t yet they are allowed to depend on the entire history of active agents $(A_1, ..., A_t)$ and not just on the t-period information. This small abuse of notation is used in order to keep notation simple. In the remainder of the paper, all objects denoted with superscript t are allowed to be history dependent.

The scoring of agent $i \in N_k \cap A_t$ is compared to the scores of all other active agents of class k in period t to form a ranking on the set $N_k \cap A_t$. The "no ties" requirement does not appear in [13] and it is made here as we sometime wish to carefully control the number of agents of each class that are rejected. In order to simplify the presentation of our mechanisms, from now on, whenever possible ties occur, we assume the existence of a tie-breaking rule instead of formally defining scoring functions with no ties.

Composition Functions: In each period a different composition function is defined. Its inputs are the bids of all previously rejected agents and its output is a subset of $\{1, ..., K\}$.

Definition 8. *In each period t, a **composition function** C^t is a function of the form $C^t : B_{N\setminus A_t} \to 2^K$.*

Multilateral Deferred-Acceptance Algorithms: Given a set of scoring functions and composition functions, an MDA algorithm is defined as follows: In each period t, the composition function $C^t(b_{N\setminus A_t})$ outputs a subset of classes. For each class $k \in C^t(b_{N\setminus A_t})$, the algorithm queries the active agents of class k and rejects the highest scoring one, according to the scoring functions $\{s^t_{k,i}\}_{i\in N_k\cap A_t}$. This means that in period t the number of agents rejected is $|C^t(b_{N\setminus A_t})|$. If $C^t(b_{N\setminus A_t}) = \emptyset$, the algorithm terminates and all active agents are declared winners. The algorithm's operation can be described in the following manner:

1. Initialize the algorithm with $A_1 = N$.
2. In each iteration $t \geq 1$, if $C^t(b_{N\setminus A_t}) = \emptyset$, stop and accept all currently active agents A_t.
3. If $C^t(b_{N\setminus A_t}) = C^t \neq \emptyset$, define:

$$A_{t+1} = A_t \setminus \bigcup_{k\in C^t} \underset{i\in N_k\cap A_t}{\operatorname{argmax}}\, s^t_{k,i}(b_i, b_{N_k\setminus A_t})$$

   and return to phase 1.

3.2 Multilateral Deferred-Acceptance Mechanisms

To complete the definition of the MDA mechanism we need to define a payment rule that accompanies the algorithm.

Definition 9. *Given an MDA algorithm and a vector of bids $b \in \Pi_{i\in N} B_i$, let $A(b)$ denote the set of winning agents, as determined by the algorithm. The **threshold payment** of a winning agent $i \in A(b)$ is defined as* $\sup\{b'_i \in B_i | i \in A(b'_i, b_{-i})\}$

Note that in the context of deferred-acceptance mechanisms, a threshold payment is the maximal bid a winning agent could have submitted that would have kept him active in all periods.

Definition 10. *An **MDA mechanism** is a mechanism which computes an allocation using an MDA algorithm and makes the corresponding threshold payments to winning agents. Losing agents pay zero.*

3.3 Main Properties of MDA Mechanisms

In this section we show that every MDA mechanism can be implemented as a DA auction. This in turn will imply that any MDA mechanism is IR, strategy-proof, WGSP and equivalent to a clock auction. The method we use is similar to the one used in [6] for the generalization of DA auctions to two-sided markets and it requires a reduction of the multilateral problem to a single-sided problem.

Proposition 11. *For every MDA mechanism there is an equivalent DA auction.*

Proof. Let $\{C^t\}$ be a set of composition functions and let $\{s^t_{k,i}\}$ be a set of scoring functions. In Appendix A we show that for every MDA mechanism there is an equivalent MDA mechanism that rejects only one agent in each period. Thus, we can assume WLOG that in every period C^t outputs only one class of agents (i.e., $|C^t(b_{N\setminus A_t})| \leq 1$ for all t and b). Define scoring functions $\{\sigma^t_i\}$ for the entire set of agents N as follows:

$$\sigma^t_i(b_{N\setminus A_t}) = \begin{cases} s^t_{k,i}(b_i, b_{N_k\setminus A_t}) & \text{if } C^t(b_{N\setminus A_t}) = k \text{ and } i \in N_k \cap A_t \\ 0 & \text{otherwise} \end{cases}$$

Recall that a DA auction rejects the highest scoring agent until all scores are zero in which case the auction terminates. Equivalence to the MDA algorithm follows from the fact that, given a bidding vector b, in each period t only agents of class $k = C^t(b_{N \setminus A_t})$ can get positive scores and thus they are the only candidates for rejection by the DA algorithm. This score is exactly the one determined by the scoring functions $\{s^t_{k,i}\}$ and so the agent that is eliminated in period t is the highest scoring agent in the set $N_k \cap A_t$, i.e., $A_{t+1} = A_t \setminus \text{argmax}_{i \in N_k \cap A_t} s^t_{k,i}(b_i, b_{N_k \setminus A_t})$, exactly as determined by the MDA algorithm. Ultimately, both mechanisms determine the same set of winners, A_T, and since this is true for all bidding vectors, both mechanisms have the same allocation rule. Furthermore, both mechanisms set threshold payments and thus they are equivalent. □

Proposition 11 establishes that MDA mechanisms inherit all the properties of DA auctions. Together with Theorem 6, we conclude that:

Corollary 12. *Any MDA mechanism is IR, strategy-proof, WGSP and equivalent to a clock auction.*

4 Applications for Supply Chains

In this section we examine some applications of MDA mechanisms to supply chains. In Section 4.1 we present a general setting of supply chains with homogeneous demand, meaning that all end consumers demand the same bundle. In Section 4.2 we present the trade reduction mechanism as defined in [2] and [3] (henceforth the Trade Reduction mechanism). In Section 4.3 we provide new insights into the Trade Reduction mechanism by implementing it as an MDA mechanism. In Section 4.4 we introduce a novel mechanism, the Modified Trade Reduction mechanism, which provides an improved outcome in terms of social welfare, compared to the Trade Reduction mechanism. In Section 4.5 the assumption of unique manufacturing technologies is relaxed.

4.1 Supply Chains

We now present a model of supply chains which follows [3] and generalizes the linear supply-chain model in [2]. We add one additional constraint on the supply chain, which was also assumed in [2].

Consider an economy with K types of items, denoted $1, .., K$. We begin by describing the production of these items.

Assumption 1 *Each product is manufactured with a **unique manufacturing technology**.*

This assumption means that all items of a specific type are manufactured in the same manner, using the same inputs.

Production in this economy can be described as a directed a-cyclical graph with K nodes representing the K different types of items. In this graph an edge (j, k) indicates that the production of item k uses item j as an input and the weight of the edge is the number of items of type j needed for production. Since any directed a-cyclical graph has a topological ordering,[12] assume WLOG that this ordering is given by the numbering of items' types. This means that the manufacturing of a type-k item makes use only of items of types $1, ..., k-1$.

This production structure allows us to characterize the production of a type-k item with a **production vector** $q^k = (q_{1,k}, ..., q_{K,k})' \in \mathbb{Z}^{k-1}_{\leq 0} \times \{1\} \times \{0\}^{K-k}$. Arguments $1, ..., k-1$ of the production vector are non-positive integers representing the quantities of inputs required for production.[13] $q_{k,k} = 1$ indicates that one unit of item k is being produced in the process. Items $k+1, ..., K$ are not involved in the production of item k, so $q_{k+1,k} = \cdots = q_{K,k} = 0$. Note that for all $j \neq k$, the weight of the edge (j, k) is $-q_{j,k}$.

We further assume that each producer in the economy can manufacture a single item. Thus, all the producers that manufacture an item of type $k \in \{1, ..., K\}$ are substitutes and will be regarded as agents of

[12] A topological ordering of a directed a-cyclical graph is an ordering of the nodes such that for every edge (j, k), the node j comes before k in that ordering.

[13] The assumption that $q_{j,k}$ for $j < k$ is an integer, rather than a real number, is without loss of generality since any amount of items can be regarded as one unit. For example, if item j is flour and all items $k > j$ are produced using amounts of flour in multiples of 0.5 kg, set one unit of item j to be 0.5 kg of flour.

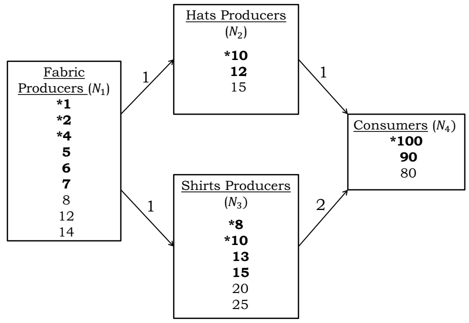

Fig. 1. An example of a simple supply chain. The optimal allocation is marked in bold and the Trade Reduction allocation is marked with asterisks.

class k. Let $c_i \in [0, \bar{c}_k]$ denote the production cost of producer i of class k. A producer's class and production vector are assumed to be common knowledge but his cost is private information.

We now turn to describe the end consumers in the economy. These agents possess no production technology but can consume all types of items. All consumers are single minded, meaning that each consumer values only one particular bundle of items and will gain zero utility from consuming any smaller bundle.

Up until now all our assumptions follow the ones made in [3]; We now add an additional assumption.

Assumption 2 ***Homogeneous demand*** - *All end consumers demand the same bundle* $d = (d_1, ..., d_K)' \in \mathbb{N}^K$ *where* d_k *is the demanded quantity of item* k.

The demanded bundle d is commonly known but consumer i's valuation for this bundle, $v_i \in [0, \bar{v}]$, is private information. We refer to consumers as agents of class $K + 1$.

Define the matrix $Q = [q^1, ..., q^K]$ and it is useful to think of it as the economy's **production matrix**. If there are μ_k producers of class k and $\mu = (\mu_1, ..., \mu_K)'$ then the supply of items is given by the vector $Q \cdot \mu$ where the k'th argument is the supply of item k. Denote by $\tilde{\mu}_k$ the number of class-k producers needed to meet the demand d of one consumer, i.e., the vector $\tilde{\mu} = (\tilde{\mu}_1, ..., \tilde{\mu}_K)$ solves $Q \cdot \tilde{\mu} = d$.[14]

For each $k = 1, ..., K + 1$, let N_k be the set of agents of class k. Assume WLOG that initially there is no excess demand of any item, i.e., that $Q \cdot \mu \geq \mu_{K+1} \cdot d$, where $\mu_k = |N_k|$ for all $k = 1, .., K + 1$ and $\mu = (\mu_1, ..., \mu_K)'$. An equivalent requirement is that $\mu \geq \mu_{K+1} \cdot \tilde{\mu}$. If this condition is not met, reject the highest bidding consumers until there is no excess demand.

We now present the concept of *procurement sets*, originally defined in [3]. This concept will be of high importance for defining the mechanisms in the following sections. A procurement set is a set of agents that contains one consumer and the minimal amount of producers needed to meet his demand. Formally:

Definition 13. *A* ***procurement set*** *is a set of agents containing one consumer and* $\tilde{\mu}_k$ *producers of class* k *for every* $1 \leq k \leq K$.

Example 14. Figure 1 depicts a simple supply chain; fabric will be referred to as item 1, hats as item 2 and shirts as item 3. The production structure is such that producing either one hat or one shirt requires one roll of fabric. This implies that the production vectors are $q^1 = (1, 0, 0)'$, $q^2 = (-1, 1, 0)'$, $q^3 = (-1, 0, 1)'$. The production matrix is therefore:

$$Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each consumer in this example demands a bundle of one hat and two shirts, i.e., $d = (0, 1, 2)'$. This implies that $\tilde{\mu} = Q^{-1} \cdot d = (3, 1, 2)'$ which means that a procurement set in this example contains one consumer, three producers of fabric, one producer of hats and two producers of shirts.

[14] Q is a unitriangular matrix with negative integers on the entries above the main diagonal. Since $d \in \mathbb{N}^K$, it can be shown that $\tilde{\mu} = Q^{-1} \cdot d$ is a vector of non-negative integers and thus appropriately represents numbers of agents.

4.2 The Trade Reduction Mechanism for Supply Chains

We now describe the Trade Reduction mechanism for the setting described above.

Definition 15. *[3] Given a set of bids, the **Trade Reduction allocation** is obtained from the optimal allocation by reducing one procurement set. The procurement set reduced is the one with the lowest value consumer and the highest cost producers out of all the procurement sets in the optimal allocation. Together with threshold payments, determined by submitted bids, this allocation rule establishes the **Trade Reduction mechanism**.*

Example 16. In Figure 1 the optimal allocation is marked in bold and the Trade Reduction allocation is marked with asterisks. In this example the threshold payments are as follows: winning fabric producers are paid 5 each, the winning hat producer is paid 12, winning shirts producers are paid 13 each and the winning consumer pays 90.

Proposition 17. *[3] The Trade Reduction mechanism is IR, strategy-proof, budget balanced and incurs the loss of the least valuable procurement set in the optimal allocation.*

4.3 The MDA Trade Reduction Mechanism

The MDA mechanism we present in this section implements the Trade Reduction allocation in a process that is equivalent to iterative rejection of procurement sets. In each iteration, the MDA algorithm examines the most valuable procurement set out of the ones that were rejected so far. The algorithm calculates the net cost of this procurement set, i.e., the sum of costs of all the producers in the set minus the value of the consumer in the set. While this net cost is strictly positive, the algorithm keeps on rejecting more procurement sets. Immediately after the mechanism rejects one procurement set with a non-negative net cost, it terminates and accepts all the currently active agents. This way only one efficient procurement set is rejected, but all the rest are accepted.

We now turn to the formal definition of the mechanism. For each $k = 1, ..., K$ let B_k be a finite set of possible bids for all producers of class k, such that $B_k \subseteq [0, \bar{c}_k]$ and $\max B_k > \bar{c}_k$. Let $B_{K+1} \subseteq [-\bar{v}, 0]$ be a finite set possible bids for all consumers, such that $\max B_{K+1} > 0$.[15]

The definition of the MDA mechanism requires defining scoring and composition functions.

Scoring Functions: Producers of each class are ranked in an ascending order of costs and consumers are ranked in a descending order of values. This way all agents are ranked in a descending order of attractiveness. Formally:

$$\begin{array}{ll} \forall\, t,\, k \in \{1, ..., K\},\, i \in A_t \cap N_k, & s^t_{k,i} = c_i \\ \forall\, t,\, i \in A_t \cap N_{K+1}, & s^t_{K+1,i} = \bar{v} - v_i \end{array} \tag{1}$$

Before turning to the definition of the composition rule, we define two auxiliary functions.

1. **The Net Cost Function** NC^t: For each period t denote by $NC^t(b_{N \setminus A_t})$ the net cost of the most valuable procurement rejected so far. Formally:

$$NC^t(b_{N \setminus A_t}) = \sum_{k=1}^{K} \sum_{l=1}^{\tilde{\mu}_k} c^t_{k,(l)} - v^t_{max} \tag{2}$$

where $c^t_{k,(l)}$ is the l'th lowest cost reported in $b_{N_k \setminus A_t}$ and v^t_{max} is the highest value reported by a rejected consumer (i.e., $-v^t_{max}$ is the minimal bid in $b_{N_{K+1} \setminus A_t}$).[16]

[15] Consumers' bid spaces are defined as subsets of $[-\bar{v}, 0]$ so we can treat all agents, producers and consumers, in a similar manner such that higher bidding agents are less attractive. Another desirable consequence of this definition is that the mechanism will determine negative monetary transfers for consumers and positive transfers for producers.

The maximal (minimal) possible bid of a producer (consumer) is set to be his highest possible cost (lowest possible value) to insure that participation is strictly preferable to non-participation (see [13]).

[16] If for any $k = 1, .., K$ it is true that $|N_k \setminus A_t| < \tilde{\mu}_k$ then set $c^t_{k,(|N_k \setminus A_t|+1)} = c^t_{k,(|N_k \setminus A_t|+2)} = ... = c^t_{k,(\tilde{\mu}_k)} = \max B_k$ and if no consumer was rejected prior to period t, set $v^t_{max} = 0$. Specifically, for $t = 1$ set $NC^1(\emptyset) = \sum_{k=1}^{K} \tilde{\mu}_k \max B_k$.

Table 1. Illustration of the MDA Trade Reduction Mechanism

t	$ES^t(b_{N\setminus A_t})$	$NC^t(b_{N\setminus A_t})$	Algorithm's Operation
1	$\emptyset$	$3\times 30+2\times 30+30-0=180$	Reject one agent of each class.
2	$\{1,3\}$		Reject one fabric producer and one shirts producer.
3	$\{1\}$		Reject one fabric producer.
4	$\emptyset$	$(14+12+8)+15+(20+25)-80=14$	Reject one agent of each class.
5	$\{1,3\}$		Reject one fabric producer and one shirts producer.
6	$\{1\}$		Reject one fabric producer.
7	$\emptyset$	$(7+6+5)+12+(15+13)-90=-32$	Terminate.

Illustration of the operation of the MDA trade reduction mechanism in the setting of Figure 1. We assume that $\max B_k = 30$ for $k = 1,2,3$ and $\max B_4 = 0$ which is needed to calculate NC^1 (see Footnote 16).

2. **The Excess Supply Function** ES^t: Let μ_k^t denote the number of active agents of class k in period t, i.e., $\mu_k^t = |N_k \cap A_t|$. The aggregate demand in period t is equal to $\mu_{K+1}^t \cdot d$ and the number of producers of class $k \in \{1, ..., K\}$ needed to meet it is $\mu_{K+1}^t \cdot \tilde{\mu}_k$. If there are more producers of class k than that, regard the class-k producers as being in excess. According to this logic, the excess supply function indicates the classes of producers that are in excess in period t. Formally:
$$ES^t(A_t) = \left\{k | 1 \le k \le K,\, \mu_k^t > \mu_{K+1}^t \cdot \tilde{\mu}_k\right\} \tag{3}$$

Composition Functions: Now we can define the composition functions, using the auxiliary functions NC^t and ES^t. For every period t, define:

$$C^t(b_{N\setminus A_t}) = \begin{cases} ES^t(A_t) & \text{if } ES^t(A_t) \neq \emptyset \\ \{1, ..., K+1\} & \text{if } ES^t(A_t) = \emptyset \text{ and } NC^t(b_{N\setminus A_t}) > 0 \\ \emptyset & \text{if } ES^t(A_t) = \emptyset \text{ and } NC^t(b_{N\setminus A_t}) \le 0 \end{cases} \tag{4}$$

In words, the algorithm first rejects excess producers, as determined by the first line in (4) (recall that in Section 4.1 we assumed that initially there is no excess demand). This is repeated until there are no excess producers, i.e., until $ES^t = \emptyset$, which means that supply equals demand. From this point, the algorithm's operation can be described as an iteration of three steps:

1. If there is no excess supply ($ES^t = \emptyset$), examine the net cost of the most valuable procurement set rejected so far, NC^t. If NC^t is non-positive - terminate (third line in (4)). Otherwise, continue to step 2.
2. Reject one agent of each class $1, ..., K+1$ (second line in (4)).
3. As long as there is excess supply ($ES^t \neq \emptyset$), reject one agent of each class $k \in ES^t$ (first line in (4)). Once $ES^t = \emptyset$, return to step 1.

It is worth noting that each time steps 1-3 are completed, one procurement set is rejected. The rejection begins with the elimination of the highest bidding agent of each class and continues with the elimination of excess supply. Since each procurement set includes only one consumer, this process is equivalent to rejecting one procurement set, and that is the highest costing active procurement set.

Example 18. Table 1 illustrates the operation of the MDA trade reduction mechanism in the setting of Figure 1. In each period, if there are no excess producers ($ES^t = \emptyset$), the net cost NC^t is calculated; if it is positive, one agent of each class is rejected. This inevitably leads to excess supply of shirts and fabric so in the following two periods two shirt producers and one fabric producer are rejected. This process is repeated twice until $ES^t = \emptyset$ and $NC^t \le 0$.

As we show in Appendix B, the MDA trade reduction mechanism is in fact equivalent to the Trade Reduction mechanism:

Proposition 19. *Consider an MDA mechanism that is defined by the scoring functions (1) and the composition functions (4). This mechanism is equivalent to the Trade Reduction mechanism (Definition 15).*

We can now use Proposition 19 (together with Proposition 12) to infer additional properties of the Trade Reduction mechanism of [2] and [3].

Corollary 20. *The Trade Reduction mechanism is WGSP and equivalent to a clock auction.*

4.4 The Modified Trade Reduction Mechanism

The class of MDA mechanisms allows considerable freedom in design while maintaining the incentive properties common to all MDA mechanisms. We use this feature to construct a new mechanism for the setting described in Section 4.1. This mechanism is a modification of the Trade Reduction mechanism described in Section 4.2 and it produces an improved outcome in terms of social welfare. The improvement is possible since the MDA trade reduction mechanism (Section 4.3) uses an inaccurate measure of the deficit - the net cost of the last rejected procurement set. This causes the mechanism to reject more trades than is actually needed in order to keep the budget balanced. The modified mechanism will use a more accurate measure of the deficit and thus will be able to waive efficient trades less frequently.

Before turning to the definition of the modified mechanism, we present the definition of t-period threshold payments. We use these payments to construct a measure of the deficit in each period.

Definition 21. *[13] For each active agent $i \in N_k \cap A_t$, the* ***t-period threshold payment*** $p^t_{k,i}(b_{N \setminus A_t})$*, is the maximal bid that would have kept i active until iteration t, holding all other bids fixed.*

First note that the mechanism's final threshold payment for a winning agent is equal to his T-period threshold payment, where T is the final period. This follows directly from the definition of threshold payments (Definition 9).

Second, note that for an active agent $i \in N_k \cap A_t$, the t-period threshold payment is determined only by bids of rejected agents of class k. This is true since agent i's bid can not affect $C^{t'}(b_{N \setminus A_t})$ for $t' < t$ and thus can not affect which classes of agents are chosen for rejection prior to t. The only effect agent i's bid has is on his ranking relative to other agents of class k. Furthermore, in cases where agents are ranked solely by their bids (as will be the case here), the t-period threshold payment of a class-k agent is equal to the bid of the last rejected agent of his class. This means that all active agents of class k have the same t-period threshold payments.

Definition 22. *Consider an MDA mechanism in which agents are ranked only by their bids. For all $t = 1, ..., T$, $k = 1, ..., K+1$, the* ***t-period threshold payments for active agents of class k*** *is:*

$$p^t_k = \begin{cases} \min\limits_{j \in N_k \setminus A_{t-1}} b_j & \text{if } N_k \setminus A_{t-1} \neq \emptyset \\ \max B_k & \text{if } N_k \setminus A_{t-1} = \emptyset \end{cases}$$

We now turn to the formal definition of the Modified Trade Reduction mechanism by defining the scoring and composition functions.

Scoring Functions: Similar to Section 4.3, producers are ranked in an ascending order of costs and consumers are ranked in a descending order of values (see (1) for the formal definition).

The composition rule is similar to the one presented in Section 4.3 with the slight difference that it rejects procurement sets according to a *lower bound* on net costs, instead of the net costs themselves. The lower bound, or **minimal net cost**, is a function of the t-period threshold payments:

$$MNC^t(b_{N \setminus A_t}) = \sum_{k=1}^{K} \tilde{\mu}_k p^t_k + p^t_{K+1} \tag{5}$$

Let the excess supply function ES^t be defined as in Section 4.3. Now define the **composition function** for each period t as follows:

$$C^t(b_{N \setminus A_t}) = \begin{cases} ES^t(A_t) & \text{if } ES^t(A_t) \neq \emptyset \\ \{1, ..., K+1\} & \text{if } ES^t(A_t) = \emptyset \text{ and } MNC^t(b_{N \setminus A_t}) > 0 \\ \emptyset & \text{if } ES^t(A_t) = \emptyset \text{ and } MNC^t(b_{N \setminus A_t}) \leq 0 \end{cases} \tag{6}$$

Definition 23. *The* ***Modified Trade Reduction mechanism*** *is an MDA mechanism defined by the scoring functions (1) and the composition functions (6).*

The Modified Trade Reduction mechanism operates as follows. First it rejects the least valuable procurement set. Fixing this allocation, the t-period threshold payments are calculated together with the implied deficit, which is proportional to MNC^t. If the deficit is non-positive, the mechanism terminates. Otherwise, the least valuable active procurement set is rejected, and so on. The critical stage of the mechanism comes after it removes enough procurement sets and reaches an efficient allocation. Then, it computes a "within-class" threshold payments for each class of agents: the value of the most valuable agent that does not win in the efficient allocation. It then checks what would happen if all agents in one procurement set paid their within-group threshold: if there is no deficit, then the mechanism outputs the efficient allocation. Otherwise, a trade reduction is performed. Note that this procedure is not equivalent to the following mechanism: run VCG if it is budget balanced, otherwise run a trade reduction. This mechanism is not truthful, as the VCG payment of an agent can be determined by agents of other classes (who therefore can manipulate the outcome). The Modified Trade Reduction mechanism uses bounds on the payments that are determined only by the agents of each class, and therefore it is strategy-proof.

Theorem 24. *The Modified Trade Reduction mechanism has the following properties:*

1. *It is IR, strategy-proof, WGSP and equivalent to a clock auction.*
2. *It is weakly budget balanced.*
3. *For every realization of values and costs, the mechanism either sets the optimal allocation or incurs the loss of the least valuable procurement set.*

Proof.

1. These properties follow from the fact that the Modified Trade reduction mechanism is an MDA mechanism.
2. See Appendix C.
3. Note that in each period, $p^t_{K+1} = -v^t_{max}$ and $p^t_k = c^t_{k,(1)}$ for all $k = 1, ..., K$. Now use the definitions of $NC^t(b_{N\setminus A_t})$ and $MNC^t(b_{N\setminus A_t})$ ((2) and (5) respectively) to get that in every period t:

$$NC^t(b_{N\setminus A_t}) = \sum_{k=1}^{K}\sum_{l=1}^{\tilde{\mu}_k} c^t_{k,(l)} - v^t_{max} \geq \sum_{k=1}^{K} \tilde{\mu}_k c^t_{k,(1)} - v^t_{max} = \sum_{k=1}^{K} \tilde{\mu}_k p^t_k + p^t_{K+1} = MNC^t(b_{N\setminus A_t}) \quad (7)$$

Since the modified mechanism terminates once $MNC^t(b_{N\setminus A_t}) \leq 0$ and the MDA trade reduction mechanism terminates once $NC^t(b_{N\setminus A_t}) \leq 0$, the former mechanism terminates (weakly) prior to the latter. Since the procurement sets are ordered in the same manner in both mechanisms, this means that the allocation determined by the modified mechanism contains the Trade Reduction allocation but is possibly larger. It therefore remains to show that the modified mechanism rejects all inefficient trades.
According to equation (7), $MNC^t(b_{N\setminus A_t})$ is a lower bound on the net costs of all previously rejected procurement sets. While $MNC^t(b_{N\setminus A_t})$ is positive, i.e., all rejected procurement sets have positive net costs, the mechanism keeps on rejecting procurement sets. Once there is one rejected procurement set with a non-positive lower bound on its net cost, the mechanism terminates. By that time all the procurement sets with positive net costs were rejected.

□

Example 25. Consider the supply chain in Figure 1. This example demonstrates a scenario where the Modified Trade Reduction mechanism is a strict improvement to the Trade Reduction mechanism.

To understand the difference between the two mechanisms it is useful to track their operation in the first four periods (Table 2). The MDA trade reduction mechanism rejects agents in period 4 because the third ranked procurement set is too expensive ($NC^4 > 0$). However, the modified mechanism examines the "within-class" payments and since these are low enough to maintain a balanced budget ($MNC^4 < 0$), the modified mechanism terminates in the fourth period.

The Modified Trade Reduction mechanism improves upon the Trade Reduction mechanism in scenarios where there is variance in the values of agents, such that the "within-class" threshold is sufficiently far from the values of the next losing agent. It follows that we need consumers to demand more than one unit of

Table 2. Comparing the MDA Trade Reduction Mechanism and the Modified Trade Reduction Mechanism

t	$ES^t(b_{N\setminus A_t})$	$NC^t(b_{N\setminus A_t})$	$MNC^t(b_{N\setminus A_t})$	Algorithm's Operation
1	$\emptyset$	$3 \times 30 + 30$ $+2 \times 30 - 0 = 180$	$3 \times 30 + 30$ $+2 \times 30 - 0 = 180$	Both mechanisms reject one agent of each class.
2	$\{1, 3\}$			Both mechanisms reject one fabric producer and one shirts producer.
3	$\{1\}$			Both mechanisms reject one fabric producer.
4	$\emptyset$	$(14 + 12 + 8) + 15$ $+(20 + 25) - 80 = 14$	$3 \times 8 + 15$ $+2 \times 20 - 80 = -1$	The Modified Trade Reduction mechanism terminates, the Trade Reduction mechanism rejects more agents.

Both mechanisms operate on the supply chain depicted in Figure 1. We assume that $\max B_k = 30$ for $k = 1, 2, 3$ and $\max B_4 = 0$ (needed to calculate NC^1 and MNC^1 - see Footnote 16 and Definition 22).

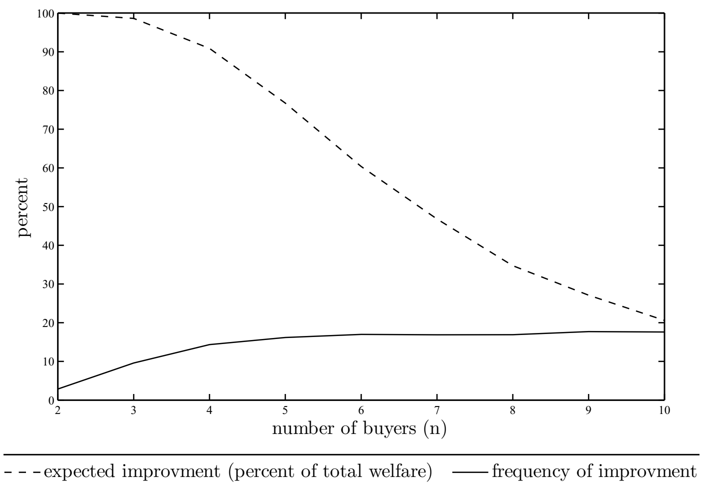

Fig. 2. Simulation results comparing the Trade Reduction mechanism and the modified mechanism: frequency of the modified mechanism improving over the Trade Reduction mechanism and the expected improvement in those cases.

some item for having a different outcome than the Trade Reduction mechanism. (Indeed, when applying the modified Trade Reduction mechanism on the two-sided market in [11], the modified mechanism identifies with the Trade Reduction Mechanism and eliminates one efficient trade.)

We ran some computer simulations of a simple supply chain network where the advantages of the modified mechanism can kick in. Our simulations considered economies with $n = 2, ..., 10$ buyers and $2n$ sellers of a homogeneous good, where each buyer is interested in a bundle of two units.[17] As can be seen in Figure 2, the modified mechanism improves upon the Trade Reduction mechanism in around 17% of the instances, even as the market grows substantially. In small economies the improvement can be up to a 100% of the overall efficiency. As expected, since our mechanism saves one valuable procurement set, the share of the overall efficiency gained decreases with market size, so our mechanism (like its predecessors) is asymptotically efficient.

4.5 Diverse Manufacturing Technologies

In this section we relax the assumption of unique manufacturing technologies (Assumption 1) and examine a setting in which part of the supply chain is modular and can be conducted using competing technologies. The modular part of the supply chain can include the production of several items and can be placed in any part of the supply chain. For convenience, we assume that the modular part includes only the production of item 2. Furthermore, we assume that item 2 can be produced by two competing technologies, yet the mechanism

[17] All values were drawn i.i.d. from the uniform distribution on $[0, 1]$.

can be easily extended to deal with any number of competing technologies. We now present this setting in detail yet some of the formal definitions, as well as the examples, are left for the appendix.

We replace Assumption 1 with the following assumption:

Assumption 3 *One item of type 2 can be manufactured with two different technologies - a and b. Items* $1, 3, 4, ..., K$ *are manufactured with unique technologies.*

Technologies a and b differ in the inputs required for production of a single item of type 2: Production with technology a involves items $a_1, ..., a_{K_a}$ and production with technology b involves items $b_1, ..., b_{K_b}$. The Production with either technology can be described as a supply chain, meaning that the production of item a_j, for $j = 1, .., K_a$ makes use only of items $a_1, ..., a_{j-1}$ and the production of item b_j, for $j = 1, .., K_b$ makes use only of items $b_1, ..., b_{j-1}$. Items a_{K_a} and b_{K_b} are perfect substitutes and may be used for consumption or for the production of items $3, 4, ..., K$. Thus they are referred to as items of type 2.

As in previous sections, producers of a type-k item for $k \in \{1, 3, 4, ..., K, a_1, ..., a_{K_a}, b_1, ..., b_{K_b}\}$, are referred to as agents of class k. Consumers are referred to as agents of class $K + 1$ and they all demand the same bundle $d \in \mathbb{N}^K$. The bundle d may contain items of types $1, .., K$ but no items of types $a_1, ..., a_{K_a-1}, b_1, ..., b_{K_b-1}$.

For each $k = 1, ..., K, a_1, ..., a_{K_a}, b_1, ..., b_{K_b}$, let N_k denote the set of agents of class k. $N_{a_{K_a}}$ and $N_{b_{K_b}}$ are the sets of agents that produce item 2 using technology a and b, respectively, and $N_2 = N_{a_{K_a}} \cup N_{b_{K_b}}$ is the set of all agents that produce items of type 2.

We maintain the notion that a procurement set is a minimal set of agents that contains one consumer and producers that meet his demand. In this setting there are two types of procurement sets: a **procurement set of type a (b)** contains one consumer and minimal number of producers of classes $1, 3, ..., K, a_1, ..., a_{K_a}$ $(1, 3, ..., K, b_1, ..., b_{K_b})$ needed to meet his demand d. (A more formal definition can be found in Appendix D.1.) We assume that meeting the demand of one consumer requires producing only one item of type 2, i.e., a procurement set contains either one producer of class a_{K_a} or one producer of class b_{K_b}.

We now turn to the definition of the Modified Trade Reduction mechanism for this setting.

Scoring functions: Producers of each class are ranked in an ascending order of costs and consumers are ranked in a descending order of values.

The following functions are used to define the composition functions (a more formal definition of these functions is given in Appendix D.1):

1. For each period t define $MNC_x^t(b_{N\setminus A_t})$, for $x = a, b$, as the total cost of a procurement set of type x, given that each agent in the set is paid his t-period threshold payment.
2. Let $ES^t(A_t) \subseteq \{1, 2, .., K, a_1, .., a_{K_a-1}, b_1, ..., b_{K_b-1}\}$ be the excess supply function that contains the classes of producers that are in excess. We assume that initially supply meets demand for each consumption good, so $ES^1(A_1) \subseteq \{a_1, .., a_{K_a-1}, b_1, ..., b_{K_b-1}\}$.

Now define the t-period **composition function** as follows:

$$C^t(b_{N\setminus A_t}) = \begin{cases} \left\{ {a_1, .., a_{K_a}, \atop 1, 3, .., K+1} \right\} & \text{if } ES^t(A_t) = \emptyset \text{ and } MNC_a^t > \max\{0, MNC_b^t\} \\ \left\{ {b_1, ..., b_{K_b}, \atop 1, 3, .., K+1} \right\} & \text{if } ES^t(A_t) = \emptyset \text{ and } MNC_b^t > \max\{0, MNC_a^t\} \\ ES^t(A_t) & \text{if } ES^t(A_t) \neq \emptyset \\ \emptyset & \text{if } ES^t(A_t) = \emptyset \text{ and } MNC_x^t \leq 0 \text{ for } x = a, b \end{cases} \tag{8}$$

Definition 26. *Consider scoring functions that rank producers in an ascending order of costs and consumers in a descending order of values. The* ***Modified Trade reduction mechanism for diverse technologies*** *is the MDA mechanism defined by these scoring functions and the composition functions (8).*

The mechanism operates by iteratively rejecting procurement sets. First, the t-period threshold payments are calculated together with the implied net cost of each type of procurement sets, $MNC_x^t, x = a, b$. If the net cost is non-positive for both types, a and b, the mechanism terminates. Otherwise, the least valuable active procurement set is rejected, and so on. Similar to the mechanism defined in Section 4.4, the Modified Trade Reduction mechanism for diverse technologies has strong incentive properties and is also budget balanced. The main difference that arises in the setting of diverse technologies is that the mechanism sometimes waives more than one procurement set from the optimal allocation. Yet it will never waive more than two procurement sets. In Appendix D.2 we prove the following proposition:

Proposition 27. *The Modified Trade reduction mechanism for diverse technologies has the following properties:*

1. *It is IR, strategy-proof, WGSP and equivalent to a clock auction.*
2. *It is weakly budget balanced.*
3. *For every realization of values and costs, the mechanism either sets the optimal allocation or incurs the loss of at most two valuable procurement sets: one of each type.*

5 Conclusion

In this paper we have considered a general multilateral market with diversified agents. The mechanism we constructed for this setting generalizes DA auctions and provides a trading platform that is individually rational, strategy-proof, weak group strategy-proof and equivalent to a clock auction.

We have also shown some applications of MDA mechanisms to settings of supply chains, but have restricted attention to supply chains with homogeneous demand. The question whether this assumption is crucial for constructing WGSP mechanisms remains open.

Another direction for future research might be to further relax the assumption of unique manufacturing technologies. [3] state that this assumption is critical for establishing the properties of the Trade Reduction mechanism, yet we have shown that it can be relaxed while maintaining even stronger incentives for agents (i.e., WGSP). Nonetheless, we have relaxed this assumption in quite a specific manner and further research in this direction might be fruitful.

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Appendix

A Completing the Proof of Proposition 11

To complete the proof of Proposition 11 we need to prove the following lemma:

Lemma 28. *For every MDA mechanism there is an equivalent MDA mechanism that rejects only one agent in each period.*

The proof will require the following definitions:

Definition 29. *Let $h_T = (A_1, ..., A_T)$ and $\tilde{h}_{\tilde{T}} = (\tilde{A}_1, ..., \tilde{A}_{\tilde{T}})$ be two histories such that $\tilde{T} \geq T$. History $\tilde{h}_{\tilde{T}}$ is **a refinement of history** h_T if:*

1. $A_T = \tilde{A}_{\tilde{T}}$ *i.e., both histories terminate with the same set of winners.*
2. $\{A_1, ..., A_T\} \subseteq \{\tilde{A}_1, ...\tilde{A}_{\tilde{T}}\}$
3. *For every* $t \leq \tilde{T}$, $|\tilde{A}_t \setminus \tilde{A}_{t-1}| = 1$ *i.e., in each period of history* $\tilde{h}_{\tilde{T}}$ *exactly one agent is rejected.*

To illustrate this notion think of a bijection that matches each period $t = 1, ..., T$ of history h_T with a period $n_t \geq t$ of history $\tilde{h}_{\tilde{T}}$ such that $A_t = \tilde{A}_{n_t}$:

$$\tilde{h}_{\tilde{T}} = \begin{pmatrix} \tilde{A}_1, & \tilde{A}_2, \ldots, \tilde{A}_{n_2}, & \tilde{A}_{n_2+1}, \ldots, \tilde{A}_{n_t}, \ldots & , \tilde{A}_{\tilde{T}} \\ (A_1 = N) & (A_2) & (A_t) & (A_T) \end{pmatrix}$$

If for every $\tilde{t} = 2, ..., \tilde{T}$, it is true that $|\tilde{A}_{\tilde{t}} \setminus \tilde{A}_{\tilde{t}-1}| = 1$, then $\tilde{h}_{\tilde{T}}$ is a refinement of h_T.

Definition 30. *Consider a set of scoring functions $\{s^t_{k,i}\}$ and a set of composition functions $\{C^t\}$.*

1. $\{C^t\}$ *is **refined** if* $|C^t(b_{N\setminus A_t})| \leq 1$ *for every vector of bids* b *and in every period* t. *An **MDA mechanism is refined** if it is defined by a set of refined composition functions.*
2. *Given the same scoring functions,* $\{s^t_{k,i}\}$, *two sets of composition functions define two different MDA mechanisms. Let* $\{C^t\}$ *and* $\{\tilde{C}^t\}$ *be two sets of composition functions and for every vector of bids* b, *let* $h_T(b)$ *and* $\tilde{h}_{\tilde{T}}(b)$ *denote the histories realized when the two mechanisms terminate, respectively.*
   $\{\tilde{C}^t\}$ *is **a refinement** of* $\{C^t\}$ *if* $\{\tilde{C}^t\}$ *is refined and for every vector of bids* b , *history* $\tilde{h}_{\tilde{T}}(b)$ *is a refinement of history* $h_T(b)$.

Note that given a set of scoring functions, if $\{\tilde{C}^t\}$ is a refinement of $\{C^t\}$, then the two mechanisms defined by $\{\tilde{C}^t\}$ and $\{C^t\}$ are equivalent. This is true since for every vector b of bids, $\tilde{h}_{\tilde{T}}(b)$ is a refinement of $h_T(b)$, so $A_T(b) = \tilde{A}_{\tilde{T}}(b)$ which means that both auctions always determine the same set of winners. In other words, both mechanism have the same allocation rule, and since they both determine threshold payments, they are equivalent.

We will now prove Lemma 28:

Proof. Let $\{C^t\}$ be a set of composition functions and let $\{s^t_{k,i}\}$ be a set of scoring functions. We need to construct a refined mechanism that is equivalent to the mechanism defined by these functions, i.e., we need to construct a refinement of $\{C^t\}$.

Let b be a vector of bids and denote by T the period in which the original algorithm terminates. Define $n_1 = 1$ and for each $t = 2, ..., T$ define $n_t = \sum_{\tau=1}^{t-1} \left|C^t(b_{N\setminus A_t})\right| + 1$.

In each period $\tilde{t} \in \{n_1, ..., n_2 - 1\} = \{1, ..., |C^1|\}$ the refined composition function $\tilde{C}^{\tilde{t}}$ will output one element from the set $C^1(\emptyset)$, starting with lowest class and continuing in an ascending order. By the end of period $n_2 - 1$ all the agents in $N \setminus A_2$ will be rejected by the new mechanism, so in period n_2 the set of active agents will be $\tilde{A}_{n_2} = A_2$. In other words, the new mechanism will reject one-by-one the highest ranked agents of class k, for all $k \in C^1(\emptyset)$.

In each period $n_2, ..., n_3 - 1$ the composition function will output one element from $C^2(b_{N\setminus A_2})$ in an ascending order of classes, and so on. By the end of period $n_T - 1$ all agents in $N \setminus A_T$ will be rejected. In period n_T the set of active agents will be $\tilde{A}_{n_T} = A_T$ and the composition function will output $\emptyset$.

The above definition is a legitimate definition of composition functions because in each period $\tilde{t} \leq n_T$ the composition function depends only on the history $\tilde{h}_{\tilde{t}}$ and the bids $b_{N\setminus \tilde{A}_{\tilde{t}}}$. To illustrate this it is convenient to present $\tilde{t}$ as $\tilde{t} = n_\tau + m$ for some $\tau = 1, ..., T$ and $m = 0, ..., |C^{\tau+1}| - 1$. Since $\tilde{C}^{\tilde{t}}(b_{N\setminus \tilde{A}_{\tilde{t}}})$ chooses the minimal class in $C^\tau(b_{N\setminus A_\tau})$ that has not yet been rejected, $\tilde{C}^{\tilde{t}}(b_{N\setminus \tilde{A}_{\tilde{t}}})$ can be defined as follows:

$$\tilde{C}^{\tilde{t}}(b_{N\setminus \tilde{A}_{\tilde{t}}}) = \min \left[C^\tau(b_{N\setminus A_\tau}) \setminus \bigcup_{j=1}^{m} \tilde{C}^{\tilde{t}-j}(b_{N\setminus \tilde{A}_{\tilde{t}-j}}) \right]$$

Thus $\tilde{C}^{\tilde{t}}(b_{N\setminus \tilde{A}_{\tilde{t}}})$ is a function of $C^\tau(b_{N\setminus A_\tau}), \tilde{C}^{\tilde{t}-m}(b_{N\setminus \tilde{A}_{\tilde{t}-m}}), ..., \tilde{C}^{\tilde{t}-1}(b_{N\setminus \tilde{A}_{\tilde{t}-1}})$. Since $\tilde{h}_{\tilde{t}-m} \subseteq ... \subseteq \tilde{h}_{\tilde{t}-1} \subseteq \tilde{h}_{\tilde{t}}$ and $N \setminus A_\tau = N \setminus \tilde{A}_{\tilde{t}-m} \subseteq ... \subseteq N \setminus \tilde{A}_{\tilde{t}-1} \subseteq N \setminus \tilde{A}_{\tilde{t}}$ it is clear that $b_{N\setminus \tilde{A}_{\tilde{t}}}$ and the history $\tilde{h}_{\tilde{t}}$ contain sufficient information for the definition of $\tilde{C}^{\tilde{t}}(b_{N\setminus \tilde{A}_{\tilde{t}}})$.[18]

B Proof of Proposition 19

The proof requires showing that the allocation includes all efficient procurement sets but one - the least valuable procurement set. We further need to show that prices are set to be the same.

It is useful to view the MDA mechanism as ranking and rejecting procurement sets. The ranking of each class of agents induces a natural partition to disjoint procurement sets. It also induces a ranking over theses sets such that the highest ranking procurement set generates the highest value from trade between its agents and the following sets are ranked in a descending order of value from trade. It is easy to see that the optimal allocation is a lower tail of this list, containing the procurement sets that have non-negative gains from trade.

The MDA algorithm examines procurement sets according to this ranking, from the bottom up, and sets an allocation which is a lower tail of that ranking. The algorithm calculates the net cost of the last rejected procurement set, denoted $NC^t(b_{N\setminus A_t})$. As long as this parameter is positive, all rejected procurement sets have positive net costs (note that $NC^t(b_{N\setminus A_t})$ is a lower bound on net costs of all rejected procurement sets), and the mechanism keeps on rejecting procurement sets. Once there is one rejected procurement set with non-positive net costs ($NC^t \leq 0$) the mechanism terminates. By that time all the procurement sets with positive net costs were rejected, meaning that all inefficient procurement sets are rejected by the MDA algorithm.

The net costs of all the accepted procurement sets are bounded from above by $NC^T(b_{N\setminus A_T})$, which is non-positive if T is the final period. This means that all accepted procurement sets have non-negative gains from trade and thus are all part of the optimal allocation. The net cost of the procurement set which was rejected last is $NC^T(b_{N\setminus A_T})$ and it is also part of the optimal allocation. Note that it is the highest ranked procurement set in the optimal allocation, i.e., the only procurement set that was rejected out of the optimal allocation is the one with highest net cost, or lowest gains from trade.

All together, the allocation rule is equivalent to that of the Trade Reduction mechanism and since both mechanisms determine threshold payments, they are equivalent.

C Proof of Theorem 24.2

The proof that the mechanism is budget balanced requires an examination of threshold payments. As aforesaid, these are equal to the bids of the last rejected agents of each class.

[18] Note the small abuse of notation: by stating that one history contains another history we mean that the former is a continuation of the latter, i.e $h_t \subseteq h_{t'}$ for $t' \geq t$, means that $h_{t'} = (h_t, A_{t+1}, ..., A_{t'})$.

The mechanism terminates with no excess supply ($ES^T(A_T) = \emptyset$) so $Q \cdot \mu^T = \mu^T_{K+1} \cdot d$ which means that:

$$\mu^T = \mu^T_{K+1} \cdot \tilde{\mu} \tag{9}$$

Note that the sum of all monetary transfers is:

$$\begin{aligned}\sum_{k=1}^{K+1} \mu^T_k p^T_k &= \sum_{k=1}^{K} \mu^T_{K+1} \tilde{\mu}_k p^T_k + \mu^T_{K+1} p^T_{K+1} \\ &= \mu^T_{K+1} \cdot \Big[\sum_{k=1}^{K} \tilde{\mu}_k p^T_k + p^T_{K+1} \Big] = \mu^T_{K+1} \cdot MNC^T(b_{N \setminus A_T}) \leq 0\end{aligned}$$

where the first equality comes from (9), the third equality comes from the definition of MNC^t given in (5) and the final inequality is because the mechanism terminated in period T so $MNC^T(b_{N\setminus A_T}) \leq 0$. □

D Diverse Manufacturing Technologies - Formal Definitions and Proofs

We begin with a formal and detailed definition of the setting described in section 4.5, including some examples. We then prove Proposition 27.

D.1 Formal Definitions and Examples

The unique manufacturing technology of an item of type $k = 1, 3, ..., K$ is described by a vector $q^k = (q^k_1, ..., q^k_K) \in \mathbb{Z}^{k-1}_{\leq 0} \times \{1\} \times \{0\}^{K-k}$ (note that item 2 is a possible input for the production of items $3, ..., K$). Let $Q = [q^{1'}, ..., q^{K'}]$ denote the production matrix of consumption goods. Let $\tilde{\mu}$ be the vector that solves $Q \cdot \tilde{\mu} = d$ and assume that $\tilde{\mu}_2 = 1$, meaning that meeting the demand of one consumer requires one producer of item 2.

Let Q^x, for $x = a, b$, denote the production matrix that represents the manufacturing of item 2 with technology x. This is a unitriangular matrix with negative integers above the main diagonal. Denote by $\tilde{\mu}^x \in \mathbb{R}^{K_x - 1} \times \{1\}$ the vector that solves $Q^x \cdot \tilde{\mu}^x = (0, ..., 0, 1)' \in \mathbb{R}^{K_x}$. We assume that for every $j = 1, ..., K_x$, the number $\tilde{\mu}^x_j$ is a natural number (rather than a general real number), meaning that $\tilde{\mu}^x_j$ may appropriately represent number of producers of class x_j. Thus, by the definition of $\tilde{\mu}^x$, a set containing $\tilde{\mu}^x_j$ producers of class x_j for all $j = 1, ..., K_x$, can produce one item of class x_{K_x}, i.e., one item of class 2.

Example 31. Figure 3 depicts a setting in which hats can be produced using two competing technologies. In this simple example, each of the competing technologies includes only one producer so $K_a = K_b = 1$, $Q^a = Q^b = 1$, $\tilde{\mu}^a = \tilde{\mu}^b = 1$. The rest of the supply chain is left unchanged relative to the one depicted in Figure 1 so $d = (0, 1, 2)'$,

$$Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\tilde{\mu} = (3, 1, 2)'$.

We can now formally define a procurement set of type $x = a, b$:

Definition 32. *A* ***procurement sets of type*** $x = a, b$ *is a set of agents containing:*

1. *One consumer*
2. $\tilde{\mu}^x_j$ *producers of item* x_j*, for each* $j = 1, .., K_x - 1$.
3. *One producer of class* K_x *(that can produce one unit of item 2)*
4. $\tilde{\mu}_k$ *producers of item* k *for each* $k = 1, 3, 4, .., K$

We can also formally define the auxiliary functions $MNC^t_x(b_{N\setminus A_t})$ and $ES^t(A_t)$:

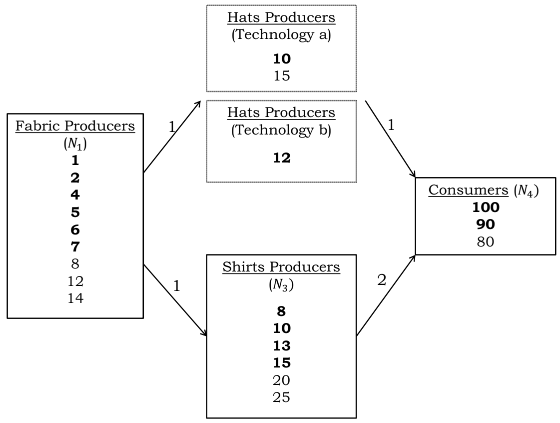

Fig. 3. Example of a simple supply chain with diverse manufacturing technologies. The Modified Trade reduction mechanism implements the optimal allocation (marked in bold).

Table 3. Illustration of the Modified Trade Reduction Mechanism for Diverse Technologies

t	$ES^t(b_{N\setminus A_t})$	$MNC_a^t(b_{N\setminus A_t})$	$MMNC_b^t(b_{N\setminus A_t})$	
1	$\emptyset$	$3 \times 30 + 30$ $+2 \times 30 - 0 = 180$	$3 \times 30 + 16$ $+2 \times 30 - 0 = 166$	Reject one fabric producer, one type-a hats producer, one shirts producer and one consumer.
2	$\{1, 3\}$			Reject one fabric producer and one shirts producer.
3	$\{1\}$			Reject one fabric producer.
4	$\emptyset$	$3 \times 8 + 15$ $+2 \times 20 - 80 = -1$	$3 \times 8 + 16$ $+2 \times 20 - 80 = 0$	Terminate.

Illustration of the MDA mechanism described in Section 4.5 for the setting depicted in Figure 3, under the assumptions that $\max B_k = 30$ for $k = a_1, 1, 3$, $\max B_{b_1} = 16$ and $\max B_4 = 0$ which is needed to calculate MNC^1 (see Definition 22).

1. For every period t and for $x = a, b$, define $MNC_x^t(b_{N\setminus A_t})$ as follows:

$$MNC_x^t(b_{N\setminus A_t}) = \tilde{\mu}_1 \cdot p_1^t + \sum_{j=1}^{K_x} \tilde{\mu}_j^x \cdot p_{x_j}^t + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^t + p_{K+1}^t \tag{10}$$

   Where p_k^t is the t-period threshold payment for an agent of class $k = x_1, ..., x_{K_x}, 1, 3, ..., K+1$.
2. For every period t, let $\mu_k^t = |N_k \cap A_t|$ denote the number of active agents of class k in period t (note that by construction $\mu_{K_a}^t + \mu_{K_b}^t = \mu_2^t$). Define the excess supply function as follows:

$$\begin{aligned} ES^t(A_t) = &\{k |\, k \in \{1, 3, ..., K\},\ \mu_k^t > \mu_{K+1}^t \cdot \tilde{\mu}_k\} \cup \\ &\{x_j |\, x \in \{a, b\},\ j \in \{1, ..., K_x - 1\},\ \mu_{x_j}^t > \mu_{x_{K_x}}^t \cdot \tilde{\mu}_j^x\} \end{aligned}$$

Example 33. In the setting depicted in Figure 3 the Modified Trade Reduction mechanism for diverse technologies implements the optimal allocation (marked in bold). The algorithm's operation is illustrated in Table 3.

D.2 Proof of Proposition 27

1. The fact that the mechanism is IR, strategy-proof, WGSP and equivalent to a clock auction follows from the fact that this is an MDA mechanism.

2. For the proof that the mechanism is budget balanced note that since the algorithm terminated in period T, $ES^T(A_T) = \emptyset$ so:
$$Q \cdot \mu^T = \mu^T_{K+1} \cdot d \tag{11}$$
$$Q^a \cdot \mu^{a,T} = (0, ..., 0, \mu^T_{a_{K_a}}), \quad Q^b \cdot \mu^{b,T} = (0, ..., 0, \mu^T_{b_{K_b}}) \tag{12}$$
where $\mu^{x,T} = (\mu^T_{x_1}, ..., \mu^T_{x_{K_x}})$.
Equations (11) and (12) give, respectively, that:
$$\mu^T = \mu^T_{K+1} \cdot \tilde{\mu} \tag{13}$$
$$\mu^{a,T} = \mu^T_{a_{K_a}} \tilde{\mu}^a, \quad \mu^{b,T} = \mu^T_{b_{K_b}} \tilde{\mu}^b \tag{14}$$
Let $\alpha = \mu^T_{a_{K_a}} / \mu^T_2$ denote the fraction of type-2 items that were manufactured with technology a. We can now rewrite the two equations in (14) by using (13) and the assumption that $\tilde{\mu}_2 = 1$ (Section 4.5):
$$\mu^{a,T} = \alpha \mu^T_2 \cdot \tilde{\mu}^a = \alpha \mu^T_{K+1} \tilde{\mu}_2 \cdot \tilde{\mu}^a = \alpha \mu^T_{K+1} \cdot \tilde{\mu}^a \tag{15}$$
$$\mu^{b,T} = (1-\alpha) \mu^T_2 \cdot \tilde{\mu}^b = (1-\alpha) \mu^T_{K+1} \tilde{\mu}_2 \cdot \tilde{\mu}^b = (1-\alpha) \mu^T_{K+1} \cdot \tilde{\mu}^b \tag{16}$$
Finally, the sum of all monetary transfers is:
$$\begin{aligned} &\mu^T_1 p^T_1 + \sum_{j=1}^{K_a} \mu^{a,T}_j p^T_{a_j} + \sum_{j=1}^{K_b} \mu^{b,T}_j p^T_{b_j} + \sum_{k=3}^{K} \mu^T_k p^T_k + \mu^T_{K+1} p^T_{K+1} \\ &= \mu^T_{K+1} \left[\tilde{\mu}^T_1 p^T_1 + \alpha \sum_{j=1}^{K_a} \tilde{\mu}^a_j p^T_{a_j} + (1-\alpha) \sum_{j=1}^{K_b} \tilde{\mu}^b_j p^T_{b_j} + \sum_{k=3}^{K} \tilde{\mu}^T_k p^T_k + p^T_{K+1} \right] \\ &= \mu^T_{K+1} [\alpha MNC^T_a(b_{N \setminus A_T}) + (1-\alpha) MNC^T_b(b_{N \setminus A_T})] \leq 0 \end{aligned}$$
Where the first equality comes from (13), (15) and (16), the second equality comes from the definition of $MNC^t_x(b_{N \setminus A_t})$ for $x = a, b$ (Equation (10)) and the final inequality follows from the fact that the algorithm terminated in period T so $MNC^T_x(b_{N \setminus A_T}) \leq 0$ for both $x = a, b$.
3. The MDA mechanism defined in Section 4.5 examines two types of procurement sets and calculates their minimal net costs (denoted $MNC^t_x(b_{N \setminus A_t})$ for $x = a, b$). As long as both types have positive bounds on net costs, a procurement set of the more expensive type is rejected (i.e., the type for which the bound $MNC^t_x(b_{N \setminus A_t})$ is higher). Once the composition rule encounters a procurement set with a non-positive bound on its net cost, it stops rejecting procurement sets of that type.[19] When both types of procurement sets have non-positive bounds on net costs, the mechanism terminates.
If T is the final period of the mechanism's operation then MNC^T_x is an upper bound on the net costs of all accepted procurement sets of type x. Since $MNC^T_x \leq 0$, all the accepted procurement sets have non-positive net costs and thus are all part of the optimal allocation.
Now, let us consider the rejected procurement sets. Define an **x-type modular set (xMS)** of the supply chain as a set of agents containing $\tilde{\mu}^x_j$ agents of class x_j, $j = 1, ..., K_x$. Define a **non-modular set** (NMS) as a set containing one consumer and $\tilde{\mu}_k$ agents of classes $k = 1, 3, ..., K+1$. These definitions give a partition of a type-x procurement set into a modular set (xMS) and a non-modular set (NMS). Denote by $xMS_{(i)}$ ($NMS_{(i)}$) the i'th cheapest (non-)modular set, in net terms, out of all the losing ones. Note that $NMS_{(i)}$ was rejected in period $T - i$.
Define a function $Cost(\cdot)$ that takes as input a procurement set and outputs the net cost of trade of that set. It suffices to show that it is possible to add to the mechanism's allocation at most one rejected procurement set of each type and still increase welfare, but that any further addition is not efficient. Equivalently, it sufficies to show that:
$$Cost(aMS_{(2)} \cup NMS_{(3)}) > 0 \tag{17}$$

[19] Note that once there is a period t in which $MNC^t_x(b_{N \setminus A_t}) \leq 0$ then for all consecutive periods $t' > t$, $MNC^{t'}_x \leq 0$ since t-period threshold payments only decrease.

$$Cost(bMS_{(2)} \cup NMS_{(3)}) > 0 \tag{18}$$

Assume WLOG that $NMS_{(1)}$ was rejected as part of a type-a procurement set, i.e., together with $aMS_{(1)}$. In that period, MNC_a^{T-1} is calculated as a lower bound on the costs of the most valuable type-a procurement set that was rejected. That procurement set is $aMS_{(2)} \cup NMS_{(2)}$ so:

$$Cost(aMS_{(2)} \cup NMS_{(2)}) \geq MNC_a^{T-1} > 0 \tag{19}$$

Where the second inequality follows from the fact that in period $T-1$ a type-a procurement was rejected (see definition of the composition functions (8)).
Since $NMS_{(3)}$ is more expensive then $NMS_{(2)}$:

$$Cost(aMS_{(2)} \cup NMS_{(3)}) > 0$$

which proves (17).
Now, let $T-i\,(i>1)$ denote the period in which $bMS_{(1)}$ was rejected, so $MNC_b^{T-i} \geq MNC_a^{T-i}$ (see definition of the composition functions (8)). Writing MNC_b^{T-i} and MNC_a^{T-i} explicitly according to (10) gives:

$$\sum_{j=1}^{K_b} \tilde{\mu}_j^b \cdot p_{b_j}^{T-i} + \tilde{\mu}_1 \cdot p_1^{T-i} + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^{T-i} + p_{K+1}^{T-i} = MNC_b^{T-i}$$
$$\geq MNC_a^{T-i} = \sum_{j=1}^{K_a} \tilde{\mu}_j^a \cdot p_{a_j}^{T-i} + \tilde{\mu}_1 \cdot p_1^{T-i} + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^{T-i} + p_{K+1}^{T-i}$$

Cancelling the common arguments on both sides of the inequality gives:

$$\sum_{j=1}^{K_b} \tilde{\mu}_j^b \cdot p_{b_j}^{T-i} \geq \sum_{j=1}^{K_a} \tilde{\mu}_j^a \cdot p_{a_j}^{T-i}$$

Since t-period threshold payments decrease with time, we can substitute $T-i$ with $T-1$ in the right hand side of the inequality:

$$\sum_{j=1}^{K_b} \tilde{\mu}_j^b \cdot p_{b_j}^{T-i} \geq \sum_{j=1}^{K_a} \tilde{\mu}_j^a \cdot p_{a_j}^{T-1}$$

Now it is possible to prove (18) by adding $\tilde{\mu}_1 \cdot p_1^{T-2} + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^{T-2} + p_{K+1}^{T-2}$ to both sides of the last inequality:

$$Cost(bMS_{(2)} \cup NMS_{(3)}) = \sum_{j=1}^{K_b} \tilde{\mu}_j^b \cdot p_{b_j}^{T-i} + \tilde{\mu}_1 \cdot p_1^{T-2} + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^{T-2} + p_{K+1}^{T-2}$$
$$\geq \sum_{j=1}^{K_a} \tilde{\mu}_j^a \cdot p_{a_j}^{T-2} + \tilde{\mu}_1 \cdot p_1^{T-2} + \sum_{k=3}^{K} \tilde{\mu}_k \cdot p_k^{T-2} + p_{K+1}^{T-2} = MNC_a^{T-2}$$
$$\geq MNC_a^{T-1} > 0$$

where the penultimate inequality is due to t-period threshold payments decreasing with time and the final inequality was explained in (19).

□