Supplement to “Posted Prices vs. Negotiations”

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Abstract

In this note we cite some results from the literature and give some proofs that are not included in our main paper. In Section 1 we give some proofs that were not included in our paper. In Section 2 we present results for distributions with light tail (i.e., with extreme-value index $\gamma = 0$). Finally, in Section 3 we cite results presented in the book by de Haan and Ferreira [1] that we use in our paper.

1 Proofs

1.1 Some Necessary Definitions

Definition 1. Let F be a distribution function with unbounded support ($F^{\leftarrow}(1) = \infty$). The *inverse quantile function* $U_F : \mathbb{R}_{>1} \to \mathbb{R}$ is

$$U_F(n) := F^{\leftarrow}\left(1 - \frac{1}{n}\right) . \tag{1}$$

For a distribution function with with bounded support ($F^{\leftarrow}(1) < \infty$) the inverse quantile function is

$$U_F(n) := F^{\leftarrow}(1) - F^{\leftarrow}\left(1 - \frac{1}{n}\right) . \tag{2}$$

Definition 2 (**Domains of attraction).** A cdf F is in the domain of attraction of the cdf G if there exists normalizing constants a_n, b_n such that for all x: $\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$.

The set $\mathcal{D}(G)$ contains of all cdf's which are in the domain of attraction of G.

We consider the following set of distributions G_γ, where $\gamma \in \mathbb{R}$ is a parameter:

$$G_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}) \ , \qquad\qquad 1+\gamma x > 0, \tag{3}$$

where for $\gamma = 0$ this is to be interpreted as $G_0 = \exp(-e^x)$.

Definition 3 (**Regularly Varying Functions).** The set RV_γ, $\gamma \in \mathbb{R}$, consists of the Lebesgue measurable functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which are eventually positive and satisfy, for all $x \in \mathbb{R}_{>0}$,

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\gamma. \tag{4}$$

We say that f is *regularly varying* with index γ.

1.2 Properties of $\widehat{U}$

We provide the proof of the following lemma that we use in our paper.

Lemma 4. *Let $U \in \mathrm{RV}_\gamma$. There exists a surjective, differentiable function $\widehat{U}$ such that*

$$\lim_{x\to\infty} \frac{\widehat{U}(x)}{U(x)} = 1 \tag{5}$$

and

$$\lim_{x\to\infty} \frac{x\widehat{U}'(x)}{\widehat{U}(x)} = \gamma. \tag{6}$$

Proof. Consider first $f \in \mathrm{RV}_0$, i.e., $\gamma = 0$. Let $t_0 > 0$ be as in Proposition B.1.5, and define $f_0(t) := \frac{\int_{t_0}^t f(s)ds}{t}$ for $t > t_0$, and $f_0(t) = 0$ for $t \le t_0$. Then, define $\hat{f}(t) := \frac{\int_0^t f_0(s)ds}{t}$. Since f is eventually positive, f_0 is eventually continuous, and thus $\hat{f}$ is eventually differentiable. We redefine $\hat{f}$ without changing the tail such that it is differentiable everywhere, and get

$$\lim_{x\to\infty} \frac{\hat{f}(x)}{f(x)} = \lim_{x\to\infty} \frac{\hat{f}(x)}{f_0(x)} \frac{f_0(x)}{f(x)} = \lim_{x\to\infty} \frac{\hat{f}(x)}{f_0(x)} \lim_{x\to\infty} \frac{f_0(x)}{f(x)} = 1, \tag{7}$$

using Proposition B.1.5, and since both these latter limits exist. Furthermore,

$$\lim_{x\to\infty} \frac{x\hat{f}'(x)}{\hat{f}(x)} = \lim_{x\to\infty} \frac{xf_0(x)}{\int_0^x f_0(x)} = 0, \tag{8}$$

since $f_0 \in \mathrm{RV}_0$ as well.

Let now $U \in \mathrm{RV}_\gamma$. Let $f(x) := U(x)x^{-\gamma}$, clearly $f \in \mathrm{RV}_0$. Let $\hat{f}$ satisfy (5) and (6), and define $\widehat{U} := x^\gamma \hat{f}(x)$. Then,

$$\lim_{x\to\infty} \frac{\widehat{U}(x)}{U(x)} = \lim_{x\to\infty} \frac{\hat{f}(x)}{f(x)} = 1 \tag{9}$$

and

$$\lim_{x\to\infty} \frac{x\widehat{U}'(x)}{\widehat{U}(x)} = \lim_{x\to\infty} \frac{x\gamma x^{\gamma-1}\hat{f}(x) + x^{\gamma+1}\hat{f}'(x)}{x^\gamma \hat{f}(x)} = \gamma + \lim_{x\to\infty} \frac{x\hat{f}'(x)}{\hat{f}(x)} = \gamma. \tag{10}$$

□

1.3 Expected Highest Order Statistic

Theorem 5. *Let $X_1, \ldots, X_n$ be i.i.d. random variables distributed according to $F \in \mathcal{D}(G_\gamma)$. Then, the expected highest order statistic $w_n^{opt} = \mathrm{E}[\max(X_1, \ldots, X_n)]$ satisfies:*

$$w_n^{opt} = U\big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)(1+o(1)), \qquad \textit{If } 0 < \gamma < 1. \tag{11}$$

$$w_n^{opt} = F^{\leftarrow}(1) - U\big((\Gamma(1-\gamma))^{\frac{1}{\gamma}} \cdot n\big)(1+o(1)), \qquad \textit{If } \gamma < 0. \tag{12}$$

Proof. Consider first the case $0 < \gamma < 1$. From [1, Theorem 5.3.2] we get that under our conditions,

$$\lim_{n\to\infty} w_n / U(n) = \int_0^\infty x d\big\{\exp(-x^{-1/\gamma})\big\} \,. \tag{13}$$

It thus only remains to compute the integral in (13). We get (first using $\int x dG = \int x G'(x) dx$, then substituting $y = x^{-1/\gamma}$)

$$\int_0^\infty x d\big\{\exp(-x^{-1/\gamma})\big\} = \int_0^\infty \frac{1}{\gamma} x^{-1/\gamma} \exp(x^{-1/\gamma}) dx = \int_0^\infty y^{-1/\gamma} \exp(y) dy = \Gamma(1-\gamma) \,. \tag{14}$$

We get $w_n = U(n)\Gamma(1-\gamma)(1+o(1)) = U(\Gamma(1-\gamma)^{1/\gamma} \cdot n)(1+o(1))$, using Proposition B.1.9(5) from [1].

In case $\gamma < 0$, we have, again from [1, Theorem 5.3.2]

$$\lim_{n\to\infty} \frac{w_n - F^{\leftarrow}(1)}{U(n)} = \int_{-\infty}^0 x d\big\{\exp(-(-x)^{1/\gamma})\big\} \,. \tag{15}$$

Proceeding as above, we get

$$\int_0^\infty x d\big\{\exp(-(-x)^{1/\gamma})\big\} = \int_0^\infty \frac{1}{\gamma} (-x)^{1/\gamma - 1} \exp(-(-x)^{1/\gamma}) \tag{16}$$

$$= \int_0^\infty y^{-1/\gamma} \exp(y) dy = \Gamma(1-\gamma) \,. \tag{17}$$

□

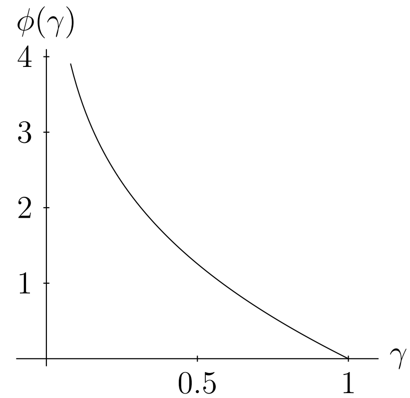

Figure 1: The function ϕ which appears in Theorem 6 in our main paper, where $\phi(\gamma)$ denotes the unique positive solution to the equation $\exp(x) = 1 + \frac{x}{\gamma}$. We remark that $\lim_{\gamma \downarrow 0} \phi(\gamma) = \infty$.

2 The special case $\gamma = 0$

In this section, we handle the special case $\gamma = 0$, $F^{\leftarrow}(1) = \infty$. The theorem claims that the expected revenue in the three mechanisms that we consider is $U(n)(1+o(1))$.

Theorem 6. *Let $F \in \mathcal{D}(G_0)$, and assume $F^{\leftarrow}(1) = \infty$. The optimal expected revenue in the mechanisms that we consider satisfies:*

$$U(n)(1 + o(1)) = r_n^{opt} \geq r_n^{disc} \geq r_n^{sym} = U(n)(1 + o(1)). \tag{18}$$

Proof. Clearly, $r_n^{opt} \geq r_n^{disc} \geq r_n^{sym}$. It remains to show that $r_n^{sym} \geq U(n)(1 + o(1))$ and $\mathrm{E}[\max(X_1, \ldots, X_n)] \leq U(n)(1 + o(1))$ which implies the rest.

We apply [1, Corollary 5.4.2] and get that for any $\epsilon > 0$,

$$\lim_{n \to \infty} \Pr\Big[\frac{\max(X_1, \ldots, X_n)}{U(n)} \in (1 - \epsilon, 1 + \epsilon)\Big] = 1, \tag{19}$$

which gives the bound $r_n^{sym} \geq U(n)(1 + o(1))$: for any $\epsilon > 0$, the bidder can post the price $U(n)(1 - \epsilon)$, which will be accepted with probability at least $1 - \epsilon$ for large enough n. This gives a revenue of at least $U(n)(1 - \epsilon)(1 - \epsilon) \geq U(n)(1 - 2\epsilon)$. Since ϵ can be made arbitrarily small we get the first bound.

We next show that $E[\max(X_1, \ldots, X_n)] = U(n)(1+o(1))$. For this we use Theorem 5.3.1 from [1]. It is well know that the integral in this theorem (the mean of the Gumbel distribution) is $\int_{-\infty}^{\infty} x dG_0(x) = \gamma_c \approx 0.577$, the Euler-Mascheroni constant. Together with Theorem 1.2.9(3) from [1] we can thus write

$$E[\max(X_1, \ldots, X_n)] = U(n) + a(n)\gamma_c(1 + o(1)) \tag{20}$$

$$= U(n)\Big(1 + \frac{a(n)}{U(n)}\gamma_c(1 + o(1))\Big) = U(n)(1 + o(1)). \tag{21}$$

□

3 Theorems from [1]

In this section we reproduce the theorems from [1] which we use throughout the paper (some of the theorems are slightly modified in order to match our notation – in particular, in [1] $U(n)$ is always defined as $F^{\leftarrow}(1-\frac{1}{n})$, whereas we use $F^{\leftarrow}(1)-F^{\leftarrow}(1-\frac{1}{n})$ in case $F^{\leftarrow}(1) < \infty$ in order to make the presentation of our results simpler).

Theorem 1.1.8. *Let F be a cdf and $b = F^{\leftarrow}(1)$ its right endpoint (possibly $b = \infty$). If $f'(x)$ exists, $f(x)$ is positive for all x in some left neighbourhood of b, and*

$$\lim_{t \uparrow b} \Big(\frac{1-F}{f}\Big)'(t) = \gamma, \tag{22}$$

then F is in the domain of attraction of G_γ.

Theorem 1.2.1. *The distribution function F is in the domain attraction of G_γ if and only if*

1. *for $\gamma > 0$: $F^{\leftarrow}(1)$ is infinite and*

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma} \tag{23}$$

   for all $x > 0$ (i.e., $(1-F) \in \mathrm{RV}_{-1/\gamma}$);

2. for $\gamma < 0$*:* $b := F^{\leftarrow}(1)$ *is finite and*

$$\lim_{t\downarrow 0} \frac{1 - F(b - tx)}{1 - F(b - t)} = x^{-1/\gamma} \tag{24}$$

for all $x > 0$*;*

3. for $\gamma = 0$*:* $F^{\leftarrow}(1)$ *can be finite or infinite and*

$$\lim_{t\uparrow F^{\leftarrow}(1)} \frac{1 - F(t + xf(t))}{1 - F(t)} = e^{-x} \tag{25}$$

for all real x *and a suitable positive function* f*.*

The function $a(t)$ is a suitable function such that $\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}$ (see Theorem 1.1.6). It is the same in Lemma 1.2.9(3) and in Theorem 5.3.1.

Lemma 1.2.9(3). *Suppose* F *is in the domain of attraction of* G_0*. Then,* $\lim_{t\to\infty} U(tx)/U(t) = 1$ *for any* $x > 0$ *and* $\lim_{t\to\infty} a(t)/U(t) = 0$*.*

Corollary 1.2.10. *For* $\gamma \neq 0$*,* F *is in the domain of attraction of* G_γ *if and only if* $U \in \mathrm{RV}_\gamma$*.*

Theorem 5.3.1. *Suppose* X_i *are i.i.d. according to* F*, and assume* F *is in the domain of attraction of* G_γ*. Let* k *be an integer with* $0 < k < 1/\max(0, \gamma)$ *and suppose* $E[|X_i|^k]$ *is finite. Then, if* $F^{\leftarrow}(1) = \infty$*:*

$$\lim_{n\to\infty} E\Big(\frac{\max(X_1, \ldots, X_n) - U(n)}{a(n)}\Big) = \int_{-\infty}^{\infty} x^k dG_\gamma(x) \ , \tag{26}$$

and if $F^{\leftarrow}(1) < \infty$*:*

$$\lim_{n\to\infty} E\Big(\frac{\max(X_1, \ldots, X_n) - (F^{\leftarrow}(1) - U(n))}{a(n)}\Big) = \int_{-\infty}^{\infty} x^k dG_\gamma(x) \ . \tag{27}$$

Theorem 5.3.2. *Suppose the conditions of Theorem 5.3.1. hold. If* $\gamma > 0$*, then*

$$\lim_{n\to\infty} E\Big[\frac{\max(X_1, \ldots, X_n)}{U(n)}\Big]^k = \int_0^{\infty} x^k d\{\exp(-x^{-1/\gamma})\} \ . \tag{28}$$

If $\gamma < 0$*, then*

$$\lim_{n\to\infty} E\Big[\frac{\max(X_1,\ldots,X_n) - F^{\leftarrow}(1)}{U(n)}\Big]^k = \int_{-\infty}^{0} x^k d\{\exp(-(-x)^{1/\gamma})\}\ . \tag{29}$$

Corollary 5.4.2. *Assume* F *is in the domain of attraction of the Gumbel distribution* $G_0(x)$*,* $F^{\leftarrow}(1) = \infty$*, and* $\epsilon > 0$*. Then,*

$$\lim_{n\to\infty} \Pr\Big[\frac{\max(X_1,\ldots,X_n)}{U(n)} \in (1-\epsilon, 1+\epsilon)\Big] = 1\ . \tag{30}$$

Theorem B.1.4 (Uniform Convergence Theorem)**.** *If* $f \in \mathrm{RV}_\gamma$ *then for any* $0 < a < b < \infty$*, then*

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\gamma \tag{31}$$

holds uniformly for $x \in [a,b]$*.*

Karamata's theorem states how a regularly varying function behaves under integration. We use it in Lemma 4 to take an arbitrary $f \in \mathrm{RV}_\gamma$ and construct a function which is differentiable and tail equivalent to f.

Theorem B.1.5 (Karamata's Theorem)**.** *Suppose* $f \in \mathrm{RV}_\gamma$*. There exists* $t_0 > 0$ *such that* $f(t)$ *is positive and locally bounded for* $t \geq t_0$*. Furthermore, if* $\gamma \geq -1$ *then*

$$\lim_{t\to\infty} \frac{tf(t)}{\int_{t_0}^{t} f(s)ds} = \gamma + 1. \tag{32}$$

If $\gamma < -1$ *then*

$$\lim_{t\to\infty} \frac{tf(t)}{\int_{t}^{\infty} f(s)ds} = -\gamma - 1. \tag{33}$$

The next theorem is due to Potter [2]. It gives bounds on the value of a regularly varying functions and their limit (one should think of it as giving a bound on $f(tx)$ in terms of $f(t)$ for any x).

Proposition B.1.9(5) (Potter)**.** *Let* $f \in \mathrm{RV}_\gamma$*. For any* $\epsilon > 0$ *there exists* t_ϵ *such that, for all* t*,*x *with* $t \geq t_\epsilon$*,* $tx \geq t_\epsilon$*:*

$$(1-\epsilon)x^\gamma \min(x^\epsilon, x^{-\epsilon}) < \frac{f(tx)}{f(t)} < (1+\epsilon)x^\gamma \max(x^\epsilon, x^{-\epsilon}). \tag{34}$$

Proposition B.1.9(10). *Suppose* $f \in \mathrm{RV}_\gamma$*,* $\gamma > 0$*, is bounded on finite intervals of* $\mathbb{R}_{>0}$*. Then* $f(f^{\leftarrow}(x)) = x(1+o(1))$ *as* $x \to \infty$*.*

In one of our proofs, it is useful to change the notation from this proposition as follows:

Lemma 7. *Suppose* $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is monotonically decreasing and satisfies* $\lim_{t \downarrow 0} \frac{f(tx)}{f(t)} = x^\gamma$ *for some* $\gamma < 0$ *and all* $x > 0$*. Let* $\overline{f}(y) = \sup\{x | f(x) \geq y\}$*. Then,* $f(\overline{f}(x)) = x(1+o(1))$ *as* $x \to 0$*.*

Proof. Define $g(x) = f(1/x)$. Then, the above relation translates to $g \in \mathrm{RV}_{-\gamma}$: $\lim_{t\to\infty} \frac{g(tx)}{g(t)} = \lim_{t\to\infty} \frac{f(1/tx)}{f(1/t)} = x^{-\gamma}$. Furthermore, $\overline{f}(y) = \sup\{x|f(x) \geq y\} = \sup\{x|g(1/x) \geq y\} = 1/\inf\{x|g(x) \geq y\} = 1/g^{\leftarrow}(y)$. We thus apply Proposition B.1.9(10) above, and get $f(\overline{f}(x)) = g(g^{\leftarrow}(x)) = x(1+o(1))$, as $x \to \infty$. □

References

[1] Laurens de Haan and Ana Ferreira. *Extreme Value Theory - An Introduction.* Springer, 2006.

[2] H.S.A. Potter. The mean value of a Dirichlet series. *Proc. London Math. Soc.*, 47:1–19, 1942.